MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J Fall 2008
Lecture 4 9/15/2008

COUNTING

Readings: [Bertsekas & Tsitsiklis], Section 1.6, and solved problems 57-58 (in 1st edition) or problems 61-62 (in 2nd edition). These notes only cover the part of the lecture that is not covered in [BT].

1 BANACH'S MATCHBOX PROBLEM

A mathematician starts the day with a full matchbox, containing n matches, in each pocket. Each time a match is needed, the mathematician reaches into a "random" pocket and takes a match out of the corresponding box. We are interested in the probability that the first time that the mathematician reaches into a pocket and finds an empty box, the other box contains exactly k matches.

Solution: The event of interest can happen in two ways:

(a) In the first $2n - k$ times, the mathematician reached n times into the right pocket, $n - k$ times into the left pocket, and then, at time $2n - k + 1$, into the right pocket.

(b) In the first $2n - k$ times, the mathematician reached n times into the left pocket, $n - k$ times into the right pocket, and then, at time $2n - k + 1$, into the left pocket.

Scenario (a) has probability

$$\binom{2n-k}{n} \cdot \frac{1}{2^{2n-k}} \cdot \frac{1}{2}.$$

Scenario (b) has the same probability. Thus, the overall probability is

$$\binom{2n-k}{n} \cdot \frac{1}{2^{2n-k}}.$$

2 MULTINOMIAL PROBABILITIES

Consider a sequence of n independent trials. At each trial, there are r possible results, $a_1, a_2, \ldots, a_r$, and the ith result is obtained ith probability p_i. What is the probability that in n trials there were exactly n_1 results equal to a_1, n_2 results equal to r_2, etc., where the n_i are given nonnegative integers that add to n?

Solution: Note that every possible outcome (n-long sequence of results) that involves n_i results equal to a_i, for all i, has the same probability, $p_1^{n_1} \cdots p_r^{n_r}$. How many such sequences are there? Any such sequence corresponds to a partition of the set $\{1, \dots n\}$ of trials into subsets of sizes $n_1, \dots, n_r$: the ith subset, of size n_i, indicates the trials at which the result was equal to a_i. Thus, using the formula for the number of partitions, the desired probability is equal to

$$\frac{n!}{n_1! \cdots n_r!} \cdot p_1^{n_1} \cdots p_r^{n_r}.$$

MIT OpenCourseWare
http://ocw.mit.edu

6.436J / 15.085J Fundamentals of Probability
Fall 2008

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.