MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Problem Set 5

Readings:
(a) Notes from Lectures 7-9.
(b) [Cinlar] Sections I.4-I.6

Exercise 1. The worker's union requests that all workers at a factory be given the day off if at least one worker has a birthday on that day. Otherwise workers agree to work 365 days a year. Management is to maximize the number of man-days worked per year. How many workers should they hire?

Solution: Management will maximize the expected number of man-days worked per year assuming that worker's birthdays are independent and identically distributed uniformly over the calender year. More specifically, given n workers, let $\{B_k \mid k = 1, \ldots, n\}$ be the birthday of the k-th worker and $D = 365$ be the number of calendar days, then, for all k and for all $d \in 1, \{\ldots, D\}$, $\mathbb{P}(B_k = d) = \frac{1}{D}$. Let $W_d(n)$ be an indicator random variable for whether or not the factory is open on day d and $W(n) = \sum_{d=1}^{D} W_d$ be the number of days worked. Then

$$\{W_d(n) = 1\} = \{\text{No worker has a birthday on day } d\}$$
$$\implies \quad \mathbb{P}(W_d(n) = 1) = \left(1 - \frac{1}{D}\right)^n,$$

and the expected number of work days is

$$E\left[W(n)\right] = \left[\sum_{d=1}^{D} W_d(n)\right] = \sum_{d=1}^{D} E\left[W_d(n)\right] = \sum_{d=1}^{D} \left(1 - \frac{1}{D}\right)^n = D\left(1 - \frac{1}{D}\right)^n.$$

Hence the expected number of man-days worked is

$$nD\left(1 - \frac{1}{D}\right)^n.$$

Consider the ratio

$$r(n) = \frac{nD\left(1 - \frac{1}{D}\right)^n}{(n-1)D\left(1 - \frac{1}{D}\right)^{n-1}} = \frac{n}{n-1}\left(1 - \frac{1}{D}\right).$$

Then $r(n) \geq 1$ for $n \leq D$ and $r(n) < 1$ for $n > D$, and the optimal number of workers is $n = D = 365$.

Exercise 2. Let $\Omega = \mathbb{Z}_+$, $\mathcal{F} = 2^\Omega$. Complete construction of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and come up with a sequence of random variables X_n which is increasing a.e., but $\mathbb{E}[X_n]$ does not converge to $\mathbb{E}[X]$, where $X = \lim_n X_n$ a.e.

Solution: Consider the probability space $(\mathbb{N}, 2^{\mathbb{N}}, \mathbb{P})$ with $\mathbb{P}(k) = \frac{6}{\pi^2}\frac{1}{k^2}$. Let $X_n(k) = -\mathbb{1}\{k \geq n\}k$. Then $X_n \leq X_{n+1}$ and for all n

$$E\left[X_n\right] = \sum_{k=n}^{\infty} -k\frac{6}{\pi^2}\frac{1}{k^2} = -\frac{6}{\pi^2}\sum_{k=n}^{\infty}\frac{1}{k} = -\infty.$$

However, $\lim_{n\to\infty} X_n = 0$. Hence

$$\lim_{n\to\infty} E\left[X_n\right] = -\infty \neq 0 = E\left[\lim_{n\to\infty} X_n\right].$$

Exercise 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probabilisty space and $X \geq 0$ a random variable. Show

$$\mathbb{E}[X] = \int_0^\infty (1 - F_X(x))dx\,,$$

where $F_X(x) = \mathbb{P}[X \leq x]$ is a CDF of X. (*Hint:* Fubini.)

Solution: We solve the problem for a more general case: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with μ σ-finite and $f \geq 0$ measurable, then

$$\int f(\omega)d\mu(\omega) = \int_0^\infty \mu[\{\omega : f(\omega) > x\}]dx,$$

Let $([0,\infty], \mathcal{B}, \lambda)$ be the Lebesgue measure space and $([0,\infty]\times\Omega, \mathcal{F}\times\mathcal{B}, \mu\times\lambda)$ the product measure space with $(\Omega, \mathcal{F}, \mu)$. Although not explicitly discussed in lecture, the construction of $([0,\infty], \mathcal{B})$ is very similar to that of $([0,\infty), \mathcal{B})$. Moreover, for a general topological space X, $\mathcal{B}(X)$ is defined as the smallest σ-algebra containing all the open sets. One way to explicitly generate the topology on $[0,\infty]$ is through the function $\tanh : [0,\infty] \to [0,1]$ with the continuous extensions $\tanh(\infty) = 1$. The open sets in $[0,\infty]$ are then the image of open sets in $[0,1]$ under $\tanh^{-1}$. As discussed in Lecture 9, the Lebesgue measure is σ-finite, and therefore the proofs for the probability measure and Fubini's theorem hold.

Consider the function $g : ([0,\infty]\times[0,\infty], \mathcal{B}\times\mathcal{B}) \to (\mathbb{R}, \mathcal{B})$

$$g(x,y) = \mathbb{1}_{(x,\infty]}(y) = \begin{cases} 1 & x < y \\ 0 & \text{else} \end{cases}.$$

Let $B \in \mathcal{B}$,

$$g^{-1}(B) = \begin{cases} [0,\infty] \times [0,\infty] & \{0,1\} \in B \\ \{x \geq y\} & \{0\} \in B \text{ and } \{1\} \notin B \\ \{x < y\} & \{1\} \in B \text{ and } \{0\} \notin B \\ \emptyset & \text{else} \end{cases}.$$

As $\{x < y\}$ is open and $\mathcal{B} \times \mathcal{B}$ contains all open sets, $g^{-1}(B) \in \mathcal{B} \times \mathcal{B}$ in all cases. Hence g is $(\mathcal{B} \times \mathcal{B}, \mathcal{B})$ measurable.

Claim: Let h_1 be $(\mathcal{F}_1, \mathcal{G}_1)$ measurable and h_2 be $(\mathcal{F}_2, \mathcal{G}_2)$ measurable. The function $h(\omega_1, \omega_2) = (h_1(\omega_1), h(\omega_2))$ is $(\mathcal{F}_1 \times \mathcal{F}_2, \mathcal{G}_1 \times \mathcal{G}_2)$ measurable.

Let

$$\mathcal{L} = \{E \in \mathcal{G}_1 \times \mathcal{G}_2 \mid h^{-1}(E) \in \mathcal{F}_1 \times \mathcal{F}_2\}.$$

Let $B_k \in \mathcal{G}_k$, $h^{-1}(B_1 \times B_2) = (h_1^{-1}(B_1) \times h_2^{-1}(B_2)) \in \mathcal{F}_1 \times \mathcal{F}_2$ by measurability of h_1 and h_2.

$$\begin{aligned} f^{-1}(\emptyset) &= \{(\omega_1, \omega_2) \mid (h_1(\omega_1), h_2(\omega_2)) \in \emptyset\} \\ &= \{(\omega_1, \omega_2) \mid h_1(\omega_1) \in \emptyset \text{ or } h_2(\omega_2) \in \emptyset\} \\ &= \emptyset \times \emptyset = \emptyset. \end{aligned}$$

Thus, $\phi \in \mathcal{L}$. Let $\{E_k\} \in \mathcal{L}$. By properties of the inverse image $h^{-1}\left(\bigcup_{k=1}^{\infty} E_k\right) = \bigcup_{k=1}^{\infty} h^{-1}(E_k) \in \mathcal{F}_1 \times \mathcal{F}_2$ and $h^{-1}(E_k^c) = \left(h^{-1}(E_k)\right)^c \in \mathcal{F}_1 \times \mathcal{F}_2$. Hence $\mathcal{L}$ is a σ-algebra containing a generating p-system for $\mathcal{G}_1 \times \mathcal{G}_2$, and by minimality, $\mathcal{L} = \mathcal{G}_1 \times \mathcal{G}_2$.

In particular, the function $h : (\mathbb{R} \times \Omega) \to (\mathbb{R} \times \mathbb{R})$ $h(x, \omega) = (x, f(\omega))$ is $(\mathcal{B} \times \mathcal{F}, \mathcal{B} \times \mathcal{B})$ measurable. Therefore, the function $(\mathbb{R} \times \Omega) \mapsto (\mathbb{R} \times \mathbb{R})$

$$\mathbb{1}_{(x,\infty]}(f(\omega)) = (g \circ h)$$

is $(\mathcal{B} \times \mathcal{F}, \mathcal{B} \times \mathcal{B})$ measurable. The iterated integrals are

$$\int_{\Omega} \int_{[0,\infty]} \mathbb{1}_{(x,\infty]}(f(\omega))\, dx\, d\mu = \int_{\Omega} f(\omega)\, d\mu,$$

and

$$\begin{aligned} \int_{[0,\infty]} \int_{\Omega} \mathbb{1}_{(x,\infty]}(f(\omega))\, d\mu\, dx &= \int_{[0,\infty]} \int_{\Omega} \mathbb{1}_{f^{-1}(x,\infty]}(\omega)\, d\mu\, dx \\ &= \int_{[0,\infty]} \mu(f^{-1}(x,\infty])\, dx. \end{aligned}$$

Moreover, as the function is nonnegative Fubini's theorem applies and these are equal

$$\int_{\Omega} f(\omega)\, d\mu = \int_{[0,\infty]} \mu(f^{-1}(x,\infty])\, dx.$$

Exercise 4. Show that for integrable f

$$\left| \int f d\mu \right| \le \int |f| d\mu.$$

Solution: By definition

$$\begin{aligned} \left| \int f\, d\mu \right| &= \left| \int f_+\, d\mu - \int f_-\, d\mu \right| \\ &\le \left| \int f_+\, d\mu \right| + \left| \int f_-\, d\mu \right| \quad \text{(Triangle Inequality)} \\ &= \int f_+\, d\mu + \int f_-\, d\mu \quad (f_+, f_- \ge 0) \\ &= \int |f|\, d\mu. \end{aligned}$$

Exercise 5 (Weird integrable functions)**.** Let $\psi(x) = \frac{1}{\sqrt{x}} \mathbb{1}_{(0,1)}(x)$ and

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} \psi(x - r_n)\,,$$

where $\{r_n\}$ is some enumeration of all rationals in $(0,1)$. Show that $F(x)$ is a measurable non-negative function with

$$\int_{[0,1]} F d\lambda < \infty\,.$$

In particular, $F(x)$ is finite almost everywhere on $[0,1]$, yet unbounded on every interval.

Solution: The function $\frac{1}{\sqrt{x}}$ is continuous on $(0,1)$ and simple functions are measurable. Therefore, as continuous functions are measurable and the product of measurable functions is measurable, for all $r \in \mathbb{Q}$, $\psi(x - r)$ is measurable. The sum of measurable functions is measurable and thus

$$f_k := \sum_{n=1}^{k} 2^{-n} \psi(x - r_n)$$

is measurable. As $\psi \geq 0$, the $\{f_k\}$ are increasing and nonnegative. In particular, for all x, the sequence $\{f_k(x)\}$ is increasing and therefore

$$\lim_{n\to\infty} f_k(x)$$

exists. By definition, for all x,

$$F(x) = \lim_{n\to\infty} f_k(x).$$

Hence $f_k \to F$ pointwise and thus F is measurable and nonnegative. By the monotone convergence theorem

$$\begin{aligned}
\int_{[0,1]} F\,d\lambda &= \lim_{k\to\infty} \int_{[0,1]} \sum_{n=1}^{k} 2^{-n}\psi(x-r_n)\,d\lambda(x) \\
&= \sum_{n=1}^{\infty} 2^{-n} \int_{[0,1]} \psi(x-r_n)\,d\lambda(x) \\
&=^{(a)} \sum_{n=1}^{\infty} 2^{-n} \int_{[-r,1-r]} \psi(x)\,d\lambda(x) \\
&= \sum_{n=1}^{\infty} 2^{-n} \int_{(0,1-r)} \frac{1}{\sqrt{x}}\,d\lambda(x) \\
&\leq \sum_{n=1}^{\infty} 2^{-n} \int_{(0,1)} \frac{1}{\sqrt{x}}\,d\lambda(x),
\end{aligned}$$

where (a) follows from a change of variables and (**??**). For all $m, n \in \mathbb{N}$

$$B_{m,n} = \left[\frac{m^2}{(n+1)^2}, \frac{m^2}{n^2}\right).$$

For a fixed m the $B_{m,n}$ are disjoint and

$$\bigcup_{n=m}^{\infty} B_{m,n} = [0,1), \tag{1}$$

and as $x^{-\frac{1}{2}}$ is decreasing

$$\frac{1}{\sqrt{x}}_{|B_{n,m}} \leq \frac{n+1}{m}. \tag{2}$$

Consider the sequence of functions

$$g_{m,k} = \sum_{n=m}^{k} \frac{n+1}{m} \mathbb{1}_{B_{m,n}}.$$

For all m, k $g_{m,k}$ is a simple functiona and thus measurable. Moreover, by (1) and disjoints of the $B_{m,n}$ these functions increase pointwise in k to a measurable function $g_m := \lim_{k\to\infty} g_{m,k}$ on $[0, 1)$. Therefore, by the montone convergence theorem

$$\begin{aligned}
\int_{[0,1)} g_m \, dx &= \sum_{n=m}^{\infty} \int_{[0,1)} \frac{n+1}{m} \mathbb{1}_{B_{m,n}} \, dx \\
&= \sum_{n=m}^{\infty} \int_{[0,1)} \frac{n+1}{m} \left(\frac{m^2}{n^2} - \frac{m^2}{(n+1)^2} \right) \\
&= \sum_{n=m}^{\infty} \int_{[0,1)} m \frac{2n+1}{n^2(n+1)} \\
&\leq 2m \sum_{n=m}^{\infty} \frac{1}{n^2} \\
&= \frac{2}{m} + 2m \sum_{n=m+1}^{\infty} \frac{1}{n^2} \\
&\leq \frac{2}{m} + 2m \int_{m}^{\infty} \frac{1}{x^2} \, dx \\
&= \frac{2}{m} + 2.
\end{aligned}$$

By (2) $\frac{1}{\sqrt{x}} \leq g_m(x)$ for all $x \in [0, 1)$. This provides

$$\int_{(0,1)} \frac{1}{\sqrt{x}} \, d\lambda(x) \leq \int_{[0,1)} \frac{1}{\sqrt{x}} \, d\lambda(x) \leq \int_{[0,1)} g_m \, d\lambda(x) \leq \frac{2}{m} + 2.$$

As this holds for all m, it holds in the limit. Therefore,

$$\int_{(0,1)} \frac{1}{\sqrt{x}} \, d\lambda(x) \leq 2.$$

Hence

$$\int_{[0,1]} F \, d\lambda \leq \sum_{n=1}^{\infty} 2 \cdot 2^{-n} < \infty$$

and F is finite almost everywhere. Moreover, as $\frac{1}{\sqrt{x}}$ is unbounded on $(0, 1)$, F is unbounded on every interval, i.e. there is a rational r_n in every interval and the function $\psi(x - r_n)$ will be unbounded.

Exercise 6. For all n, let g_n and g be measurable functions. Suppose that $g_n \uparrow g$ and that $\int g_1^- \, d\mu < \infty$. Prove that $\int g_n \, d\mu \uparrow \int g \, d\mu$.

Solution: Let us decompose g and each of the g_n into a pair of nonnegative functions g^- and g^+, and g_n^- and g_n^+, such that $g = g^+ - g^-$ and $g_n = g_n^+ - g_n^-$. Since $g_n \uparrow g$, then we have that $g_n + g_1^-$ are nonnegative functions such that $g_n + g_1^- \uparrow g + g_1^-$. Then, using the fact that $\int g_1^- \, d\mu < \infty$ and the MCT, we have

$$\begin{aligned}
\int g_n d\mu &= \int g_n + g_1^- - g_1^- d\mu \\
&= \int g_n + g_1^- d\mu - \int g_1^- d\mu \\
&\uparrow \int g + g_1^- d\mu - \int g_1^- d\mu \\
&= \int g + g_1^- - g_1^- d\mu \\
&= \int g d\mu.
\end{aligned}$$

Exercise 7. (Differentiating under the integral sign)
Let $g : \mathbb{R}^2 \mapsto \mathbb{R}$ be a continuous function of two variables s and x. Furthermore, assume that the derivative $g'(s,x) = (\partial g/\partial s)$ exists for every s and x, is jointly measurable in (s,x) and is a continuous function of s for any fixed x. Assume $|g'(s,x)| \leq c$ for all s, x.

Let X be a random variable. Show that

$$\frac{\partial}{\partial s}\mathbb{E}[g(s,X)] = \mathbb{E}\left[\frac{\partial g}{\partial s}(s,X)\right].$$

Note: You can use the fact from elementary calculus that under our assumptions, $g(s,x) = g(0,x) + \int_0^s \frac{\partial g}{\partial s}(u,x)du$ for all x.

Solution: Taking expectation on both sides of the given identity,

$$\mathbb{E}[g(s,X)] = \mathbb{E}[g(0,X)] + \mathbb{E}\int_0^s \frac{\partial g}{\partial s}(u,x)du$$

Since the Lebesgue measure on $\mathbb{R}$ is σ-finite and $|\frac{\partial g}{\partial s}(u,x)| \leq c$, and c is integrable over $[0,s]$, Fubini theorem yields

$$\mathbb{E}\int_0^s \frac{\partial g}{\partial s}(u,x)du = \int_0^s \mathbb{E}\frac{\partial g}{\partial s}(u,x)du.$$

As a result, $\mathbb{E}[g(s,X)] = \mathbb{E}[g(0,X)] + \int_0^s \mathbb{E}\frac{\partial g}{\partial s}(u,x)du$ is differentiable with derivative at s equal to $\mathbb{E}\frac{\partial g}{\partial s}(s,x)$.