In-class quiz 1

- You have 110 minutes to complete the quiz.

- This is a closed-book quiz, except that three $8.5'' \times 11''$ sheets of notes are allowed.

- Calculators are allowed (provided that erasable memory is cleared), but will probably not be useful.

- There are three problems on the quiz.

- The problems are not necessarily in order of difficulty.

- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.

- If we can't read it, we can't grade it.

- If you don't understand a problem, please ask.

Problem Q-1 (20 points)

The source code construction used to demonstrate the entropy bounds uses a code word length $l_i = \lceil -\log p_i \rceil$; this problem illustrates that the optimal code might deviate greatly from this choice of lengths.

(a) Give an example of a Huffman code in which l_i is one but $\log(1/p_i)$ is arbitrarily large.

(b) Find an example of a Huffman code with 7 code words in which one code word has length 6 but $\lceil \log(1/p_i) \rceil = 4$.

(c) Explain how this can be generalized to Huffman codes in which $l_i - \lceil \log(1/p_i) \rceil$ is arbitrarily large for an least one code word.

Problem Q-2 (35 points)

Recall that the AEP property, when applied to a string of n iid chance variables $X_1, \ldots, X_n$ shows that the typical set T_ε,

$$T_\varepsilon = \left\{ \mathbf{x}^n : \ 2^{-n(H(X)+\varepsilon)} < p_{\mathbf{X}^n}(\mathbf{x}^n) < 2^{-n(H(X)-\varepsilon)} \right\},$$

has a probability greater than $1 - \delta$, for any desired $\varepsilon > 0$ and $\delta > 0$ for large enough n. Here we investigate the behavior of a Huffman code on such a string for a given ε and δ.

(a) Show that a Huffman code can be rearranged, with no loss in expected length, into a code for which the "binary decimal" numbers associated with the code words are increasing with decreasing code word probabilities.

(b) Give an intuitive explanation for why the most probable code words, $i.e.$, those with $\left\{ p_{\mathbf{X}^n}(\mathbf{x}^n) \geq 2^{-n(H(X)-\varepsilon)} \right\}$ are not viewed as typical.

(c) Assme that there are both intermediate nodes and leaf nodes at some given length l. Prove that each code word of length l has a probability $p \geq q_l/2$ where q_l is the maximum of the probabilities of the intermediate nodes of length l.

(d) Let m be the shortest length for which leaf nodes exist (you may assume that all such leaf nodes correspond to atypical n-tuples). Let M_m be the number of leaf nodes of length m. Let $\delta_m \leq \delta$ be the sum of the probabilities of these atypical leaf nodes. Find a lower bound to δ_m in terms of q_m (the maximum of the probabilities of the intermediate nodes of length m) and M_m. Hint: Use part (c).

(e) Find a lower bound to q_m in terms of δ_m and M_m. Hint: The sum of the probabilities of the intermediate nodes plus leaf nodes at length m must be one.

(f) Let $\beta_m = M_m/2^m$ be the fraction of nodes at length m that are leaf nodes. Show that

$$\frac{\beta_m}{1 - \beta_m} \leq \frac{2\delta_m}{1 - \delta_m}$$

Note: The point of this problem is to show that the part of the Huffman code tree used by the aypical nodes is negligible; doing the whole job, however, would have made the problem too long.

Problem Q-3 (30 points)

Consider choosing a 2D vector quantizer over a very large region A by choosing a very large number M of quantization points, $V_1, V_2, \ldots, V_M$ at random, using a uniform probability density over A for each quantization point, and choosing each point independently. This might seem foolish, but it is a theoretically important tool for high dimensional spaces, and is somewhat easier to analyze for the 2D case. Consider a source which produces iid outputs, also with a uniform probability density.

(a) Let U be a source output. Find the probability that the distance from U to V_1 exceeds some given number r. Ignore edge effects throughout, *i.e.*, assume that the sample value of U is more than r away from the boundary of the region A.

(b) Find the probability that the distance from U to each of the $\{V_j\}$ (*i.e.*, to the closest of the $\{V_j\}$) exceeds r.

(c) Assume that r^2/A is extremely small and approximate the probability in (b) as $e^{-Mg(r)}$ for the appropriate function $g(r)$.

(d) Let R be the error when the source output is represented as the closest quantization point. Express the distribution function of the random variable R in terms of your answer to c.

(e) Find the mean square error. The mean square error here is averaged over both the source output and the random choice of quantizer points. Compare your result with that of a quantizer using a square of quantization regions.

Problem Q-4 (15 points)

Assume that $u(t)$ is a finite-energy complex-valued function. Let $\{\theta_k(t); 1 < k < \infty\}$ be a set of orthogonal waveforms and assume that $u(t)$ can be expanded in the orthogonal series

$$u(t) = \sum_{k=1}^{\infty} u_k \theta_k(t)$$

The set of waveforms satisfy

$$\int_{-\infty}^{\infty} \theta_k(t)\theta_j^*(t)\, dt = \langle \boldsymbol{\theta}_k, \boldsymbol{\theta}_j \rangle = \begin{cases} 0 & \text{for} \quad k \neq j \\ A_j & \text{for} \quad k = j \end{cases}$$

where $\{A_j\}$ is an arbitrary set of non-negative numbers.

(a) Express the coefficients $\{u_k\}$ as inner products involving $u(t)$, $\{\boldsymbol{\theta}_k\}$, and $\{A_k\}$.

(b) Find the energy $\|\mathbf{u}\|^2 = \int_{-\infty}^{\infty} |u(t)|^2 dt$ in the simplest form you can in terms of $\{\boldsymbol{\theta}_k\}$, and $\{A_k\}$