Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 7 Solutions
Fall 2016

Problem 7.1
Here we shall develop the quantum state for a single-mode field that prevails when that mode is in thermal equilibrium at temperature T K.

(a) Let's try to maximize the objective function,

$$F(\{P_n\}, \lambda_1, \lambda_2) \equiv -\sum_{n=0}^{\infty} P_n \ln(P_n) + \lambda_1 \left(1 - \sum_{n=0}^{\infty} P_n\right) + \lambda_2 \left(\mathcal{E} - \sum_{n=0}^{\infty} \hbar\omega n P_n\right),$$

by straightforward multivariable calculus. First we look for the stationary points of this function, i.e., the $\{P_n\}, \lambda_1, \lambda_2$ that satisfy,

$$\begin{aligned}
\frac{\partial F(\{P_n\}, \lambda_1, \lambda_2)}{\partial P_n} &= 0, \quad \text{for } n = 0, 1, 2, \ldots, \\
\frac{\partial F(\{P_n\}, \lambda_1, \lambda_2)}{\partial \lambda_1} &= 0, \\
\frac{\partial F(\{P_n\}, \lambda_1, \lambda_2)}{\partial \lambda_2} &= 0.
\end{aligned}$$

These stationary points are extrema—local maxima or local minima or saddle points—of the objective function. We then check the matrix of second derivatives of the objective function at these stationary points. Stationary points for which this matrix is negative semi-definite are local maxima. If there is only one such point, it is also the global maximum. However, if there are multiple stationary points at which the second-derivative matrix is negative semi-definite we must evaluate the objective function at all of these local maxima to determine the global maximum.

Requiring that the partial derivatives of the objective function with respect to the Lagrange multipliers be zero enforces the constraints on our entropy maximization, viz., when these partial derivatives are zero we have that

$$\begin{aligned}
\frac{\partial F(\{P_n\}, \lambda_1, \lambda_2)}{\partial \lambda_1} &= 1 - \sum_{n=0}^{\infty} P_n = 0, \\
\frac{\partial F(\{P_n\}, \lambda_1, \lambda_2)}{\partial \lambda_2} &= \mathcal{E} - \sum_{n=0}^{\infty} \hbar\omega n P_n = 0.
\end{aligned}$$

Adjoining these constraints to the rest of the maximization procedure described above makes it clear that unconstrained maximization of $F(\{P_n\}, \lambda_1, \lambda_2)$ is equivalent to maximizing $S(\{P_n\})$ subject to the constraints $\sum_{n=0}^{\infty} P_n = 1$ and $\sum_{n=0}^{\infty} \hbar\omega n P_n = \mathcal{E}$. In particular, suppose that the multivariable-calculus unconstrained maximization of F were not equivalent to the constrained maximization of S. Then, for any λ_1, λ_2, substituting the $\{P_n\}$ obtained from the constrained maximization of S into $F(\{P_n\}, \lambda_1, \lambda_1)$ would lead a value *larger* than the global maximum of F as found from multivariable calculus, which constitutes a contradiction.

(b) We have that,

$$\frac{\partial F(\{P_n\}, \lambda_1, \lambda_2)}{\partial P_n} = -\ln(P_n) - 1 - \lambda_1 - n\hbar\omega\lambda_2 = 0, \quad \text{for } n = 0, 1, 2, \ldots,$$

for the first equation in (a), whence,

$$P_n = e^{-(1+\lambda_1+n\hbar\omega\lambda_2)}, \quad \text{for } n = 0, 1, 2, \ldots,$$

where λ_1 and λ_2 must be used to ensure that $\sum_{n=0}^{\infty} P_n = 1$ and $\sum_{n=0}^{\infty} \hbar\omega n P_n = \mathcal{E}$ prevail. The only non-zero second derivatives for this problem are,

$$\frac{\partial^2 F(\{P_n\}, \lambda_1, \lambda_2)}{\partial P_n^2} = -\frac{1}{P_n} < 0, \quad \text{for } P_n > 0,$$

so the preceding $\{P_n\}$ is indeed the global maximum that we seek. (Strictly speaking, we should also check that the $\{P_n\}$ are all positive. We'll do that check in (f), below.)

(c) Using the sum formula for the geometric series, we can sum the $\{P_n\}$ result from (b) to get,

$$\sum_{n=0}^{\infty} P_n = e^{-(1+\lambda_1)} \sum_{n=0}^{\infty} e^{-n\hbar\omega\lambda_2} = \frac{e^{-(1+\lambda_1)}}{1 - e^{-\hbar\omega\lambda_2}} = 1.$$

This result allows us to eliminate λ_1 from the result of (b), viz., the entropy-maximizing probability distribution is,

$$P_n = (1 - e^{-\hbar\omega\lambda_2})e^{-n\hbar\omega\lambda_2}, \quad \text{for } n = 0, 1, 2, \ldots,$$

where λ_2 must be chosen to ensure that $\sum_{n=0}^{\infty} \hbar\omega n P_n = \mathcal{E}$ prevails.

(d) We have that,

$$\sum_{n=0}^{\infty} n\hbar\omega P_n = (1 - e^{-\hbar\omega\lambda_2}) \sum_{n=0}^{\infty} n\hbar\omega e^{-n\hbar\omega\lambda_2} = \mathcal{E}.$$

Differentiation of the sum formula for the geometric series yields,

$$\frac{\hbar\omega e^{-\hbar\omega\lambda_2}}{(1-e^{-\hbar\omega\lambda_2})^2} = \frac{\partial}{\partial\lambda_2}\left(\frac{1}{1-e^{-\hbar\omega\lambda_2}}\right) = \frac{\partial}{\partial\lambda_2}\left(\sum_{n=0}^{\infty} e^{-n\hbar\omega\lambda_2}\right) = \sum_{n=0}^{\infty} n\hbar\omega e^{-n\hbar\omega\lambda_2}$$

Combining the two preceding equations, we get,

$$\mathcal{E} = \frac{\hbar\omega e^{-\hbar\omega\lambda_2}}{1-e^{-\hbar\omega\lambda_2}} = \frac{\hbar\omega}{e^{\hbar\omega\lambda_2}-1}$$

for the field mode's average energy above its ground state, i.e., above the zero-point energy, $\hbar\omega/2$.

(e) With $\lambda_2 = 1/kT$, we now have,

$$N \equiv \mathcal{E}/\hbar\omega = \frac{1}{e^{\hbar\omega/kT}-1},$$

from (d). At $\lambda = 1.55\,\mu$m wavelength, we have that $\hbar\omega = 1.28\times10^{-19}$ joules. At $T = 290\,$K, we have that $kT = 4.00\times10^{-21}$ joules. Thus, we get,

$$N = \frac{1}{e^{32.0}-1} = 1.32\times10^{-14}.$$

Because the photon energy $\hbar\omega$ at the fiber communication wavelength is so much higher than the room temperature thermal-fluctuation energy kT, the average number of thermal-noise photons in the field mode is *extremely* small.

(f) From (d) we have that,

$$e^{\hbar\omega/kT} = \frac{N+1}{N}.$$

Then, using this result and $\lambda_2 = 1/kT$ in (c), we find,

$$P_n = \left(1-\frac{N}{N+1}\right)\left(\frac{N}{N+1}\right)^n = \frac{N^n}{(N+1)^{n+1}}, \quad \text{for } n = 0,1,2,\ldots,$$

as desired.

Problem 7.2

Here we shall explore what happens when a Bogoliubov (squeezing) transformation is applied to a field mode that is in thermal equilibrium at temperature $T\,$K.

(a) The mean values of the *input*-mode's quadrature measurements are easily found. We know that these mean values are zero when the field is in a number state. Because the thermal equilibrium state is diagonal in the number basis, i.e., it is a random mixture of number states with the Bose-Einstein probability distribution, the unconditional means of the quadrature measurements are found by averaging their values when they are in the number state $|n\rangle$ over the Bose-Einstein distribution. Clearly, this procedure yields,

$$\langle \hat{a}_{IN_1} \rangle = 0 \quad \text{and} \quad \langle \hat{a}_{IN_2} \rangle = 0.$$

Now, because μ and ν are both real valued, the Bogoliubov transformation that generates $\hat{a}_{OUT}$ from $\hat{a}_{IN}$ implies,

$$\hat{a}_{OUT_1} = (\mu + \nu)\hat{a}_{IN_1} \quad \text{and} \quad \hat{a}_{OUT_2} = (\mu - \nu)\hat{a}_{IN_2} \tag{1}$$

Averaging these equations then gives us,

$$\langle \hat{a}_{OUT_1} \rangle = 0 \quad \text{and} \quad \langle \hat{a}_{OUT_2} \rangle = 0.$$

(b) Because the quadrature measurements are zero-mean, their variances equal their mean-squared values. This is true for *both* the input and the output modes of the Bogoliubov transformation. Now, when the input mode is in the number state $|n\rangle$, we know that,

$$\langle \hat{a}^2_{IN_1} \rangle = \langle \hat{a}^2_{IN_2} \rangle = \frac{2n+1}{4}.$$

Averaging these results over the Bose-Einstein distribution gives the unconditional mean squares,

$$\langle \hat{a}^2_{IN_1} \rangle = \langle \hat{a}^2_{IN_2} \rangle = \frac{2N+1}{4},$$

where $N = 1/(e^{\hbar\omega/kT} - 1)$ is the average photon number in the input mode. Now, using Eq. (1), we get,

$$\begin{aligned}
\langle \Delta\hat{a}^2_{OUT_1} \rangle &= \langle \hat{a}^2_{OUT_1} \rangle = (\mu+\nu)^2 \langle \hat{a}^2_{IN_1} \rangle = (\mu+\nu)^2 \frac{2N+1}{4} \\
\langle \Delta\hat{a}^2_{OUT_2} \rangle &= \langle \hat{a}^2_{OUT_2} \rangle = (\mu-\nu)^2 \langle \hat{a}^2_{IN_2} \rangle = (\mu-\nu)^2 \frac{2N+1}{4}
\end{aligned}$$

This result shows that the thermal noise in the input mode is squeezed—because μ, ν are positive—in the 2-quadrature of the output mode, and anti-squeezed (stretched?) in the 1-quadrature of the output mode. However, because $N > 0$ for $T > 0$, we see that the output state is *never* a minimum uncertainty-product state, i.e.,

$$\langle \Delta\hat{a}^2_{OUT_1} \rangle \langle \Delta\hat{a}^2_{OUT_2} \rangle = \left(\frac{2N+1}{4} \right)^2 > \frac{1}{16}.$$

Problem 7.3
Here we shall develop the semiclassical treatment for photodetection of a chaotic field, i.e., one whose classical complex-amplitude is a zero-mean, complex-valued Gaussian random variable with statistically independent, identically distributed real and imaginary parts. Because of what we have already done in Problem Set 1, the present problem is trivial.

(a) From Problem 1.4, we know that $r \equiv \sqrt{a_{S_1}^2 + a_{S_2}^2}$ is Rayleigh distributed, with probability density function

$$p_r(R) = \begin{cases} \frac{2R}{N} e^{-R^2/N}, & \text{for } 0 \leq R, \\ 0, & \text{otherwise.} \end{cases}$$

From Problem 1.3, we know that $y \equiv r^2 = |a_S|^2$ is exponentially distributed, with probability density function,

$$p_y(Y) = \begin{cases} \frac{e^{-Y/N}}{N}, & \text{for } Y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) From the results of (a) and Problem 1.5(a) we immediately find that the photon-count probability distribution is Bose-Einstein:

$$P_n = \frac{N^n}{(N+1)^{n+1}}, \quad \text{for } n = 0, 1, 2, \ldots$$

(c) From Problem 1.5(c) we now get,

$$\langle N_S \rangle = N \quad \text{and} \quad \langle \Delta N_S^2 \rangle = N + N^2,$$

where the first term in the variance expression is the shot noise and the second term in the variance expression is the excess noise.

Problem 7.4
Here we shall examine the semiclassical photodetection statistics for the superposition of a coherent field and a chaotic field.

(a) Given knowledge of a_S, we have the

$$\langle N_S \rangle = |a_S|^2 = \alpha_S^2 + 2\alpha_S n_{S_1} + |n_S|^2,$$

where we have used the fact that α_S is real valued. Averaging over the statistics of n_S now gives us the unconditional mean value,

$$\langle N_S \rangle = \alpha_S^2 + 2\alpha_S \langle n_{S_1} \rangle + \langle |n_S|^2 \rangle = \alpha_S^2 + \langle (n_{S_1}^2 + n_{S_2}^2) \rangle = \alpha_S^2 + N.$$

(b) The conditional mean-square of N_S, given knowledge of a_S, is

$$\langle N_S^2 \rangle = |a_S|^2 + |a_S|^4 = \alpha_S^2 + 2\alpha_S n_{S_1} + |n_S|^2 + (\alpha_S^2 + 2\alpha_S n_{S_1} + |n_S|^2)^2,$$

from the conditional distribution's being Poisson with mean $|a_S|^2$. Averaging this result over the statistics of n_S provides the unconditional mean square,

$$\langle N_S^2 \rangle = \alpha_S^2 + N + \langle (\alpha_S^2 + 2\alpha_S n_{S_1} + |n_S|^2)^2 \rangle$$

To evaluate the second term on the right, we square out and average:

$$\begin{aligned}
&\langle (\alpha_S^2 + 2\alpha_S n_{S_1} + |n_S|^2)^2 \rangle \\
&= \alpha_S^4 + 4\alpha_S^3 \langle n_{S_1} \rangle + 4\alpha_S^2 \langle n_{S_1}^2 \rangle + 4\alpha_S \langle n_{S_1}(n_{S_1}^2 + n_{S_2}^2) \rangle + 2\alpha_S^2 \langle |n_S|^2 \rangle + \langle |n_S|^4 \rangle \\
&= \alpha_S^4 + 4\alpha_S^2 N + 2N^2,
\end{aligned}$$

where we have used the statistical independence of n_{S_1} and n_{S_2}, the fact that the third moment of a zero-mean Gaussian random variable is zero, and the complex-Gaussian moment factoring result $\langle |n_S|^4 \rangle = 2\langle |n_S|^2 \rangle^2$. We now have the final result for the unconditional mean square:

$$\langle N_S^2 \rangle = \alpha_S^2 + N + \alpha_S^4 + 4\alpha_S^2 N + 2N^2 = \alpha_S^2 + N + (\alpha_S^2 + N)^2 + 2\alpha_S^2 N + N^2.$$

(c) The variance is the mean square minus the square of the mean, so,

$$\langle \Delta N_S^2 \rangle = (\alpha_S^2 + N) + (2\alpha_S^2 N + N^2).$$

In this variance expression, $(\alpha_S^2 + N)$ is the shot-noise term and $(2\alpha_S^2 N + N^2)$ is the excess noise term.

(d) We have that the unconditional photon-counting probability distribution obeys,

$$\begin{aligned}
\Pr(N_S = n) &= \int d^2 a_S \, \Pr(\, N_S = n \mid a_S \,) p(a_S) \\
&= \int d^2 a_S \, \frac{|a_S|^{2n}}{n!} e^{-|a_S|^2} \frac{e^{-|a_S - \alpha_S|^2/N}}{\pi N} \\
&= \frac{e^{-\alpha_S^2/N}}{n! N} \int_0^\infty dr\, 2r^{2n+1} e^{-r^2(N+1)/N} \int_0^{2\pi} d\phi \, \frac{e^{2\alpha_S r \cos(\phi)/N}}{2\pi} \\
&= \frac{e^{-\alpha_S^2/N}}{n! N} \int_0^\infty dr\, 2r^{2n+1} e^{-r^2(N+1)/N} I_0(2\alpha_S r/N) \\
&= \frac{N^n}{(N+1)^{n+1}} e^{-\alpha_S^2/(N+1)} L_n\!\left(-\frac{\alpha_S^2}{N(N+1)} \right).
\end{aligned}$$

The first equality follows from basic probability theory: $p(a_S)$ is the probability density function for the complex-valued random variable a_S, i.e., the joint probability density function for the real-valued random variables a_{S_1} and a_{S_2}, and the notational definition

$$\int d^2 a_S \equiv \int_{-\infty}^{\infty} da_{S_1} \int_{-\infty}^{\infty} da_{S_2}.$$

The second equality uses the Poisson distribution for the conditional photon counting probabilities, given a_S, and the joint Gaussian distribution for a_{S_1} and a_{S_2}. The third equality follows from introduction of polar coordinates, $a_S = re^{j\phi}$ with $r \geq 0$. The fourth equality follows from the following definite integral,

$$\int_0^{2\pi} d\phi\, \frac{e^{\beta \cos(\phi)}}{2\pi} = I_0(2\beta),$$

as found in standard integral tables. The last equality follows from the definite integral that was given on the problem set.