Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Lecture Number 22
Fall 2016

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Date: Thursday, December 1, 2016

Introduction

Last time we established the quantum version of coupled-mode theory for spontaneous parametric downconversion (SPDC). We exhibited the exact solutions for the output signal and idler beams, their jointly Gaussian state characterization when the input beams are in their vacuum states, and the low-gain regime approximations for the correlation functions that characterize that state. We also introduced the lumped-element coupled-mode equations for the optical parametric amplifier (OPA), presented their solutions, described their jointly Gaussian state when the signal and idler inputs are unexcited, and showed that the OPA produced quadrature-noise squeezing. Today, we shall finish our survey of the nonclassical signatures produced by $\chi^{(2)}$ interactions by considering Hong-Ou-Mandel interferometry, the generation of polarization-entangled photon pairs from SPDC, and the photon-twins behavior of the signal and idler beams from an OPA. Along the way we will learn about quantum interference and photon indistinguishability.

Quantum Interference

Let us get started with a simple single-mode description in order to introduce quantum interference. Consider the 50-50 beam splitter arrangement shown on slide 3. Here, the only excited modes at the input ports are the co-polarized, pure-tone, plane-wave pulses $\hat{a}_{S_{\text{in}}} e^{-j\omega_0 t}/\sqrt{T}$ and $\hat{a}_{I_{\text{in}}} e^{-j\omega_0 t}/\sqrt{T}$, for $0 \leq t \leq T$. The resulting excited modes at the beam splitter's output then have annihilation operators given by[1]

$$\hat{a}_{S_{\text{out}}} = \frac{j\hat{a}_{S_{\text{in}}} - \hat{a}_{I_{\text{in}}}}{\sqrt{2}} \quad \text{and} \quad \hat{a}_{I_{\text{out}}} = \frac{-\hat{a}_{S_{\text{in}}} + j\hat{a}_{I_{\text{in}}}}{\sqrt{2}}. \tag{1}$$

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[1]The reader should check that this is indeed a unitary transformation and that it conserves energy and commutator brackets. It differs from the 50-50 beam splitter relation, $\hat{a}_{S_{\text{out}}} = (\hat{a}_{S_{\text{in}}} + \hat{a}_{I_{\text{in}}})/\sqrt{2}$ and $\hat{a}_{I_{\text{in}}} = (\hat{a}_{S_{\text{in}}} - \hat{a}_{I_{\text{in}}})/\sqrt{2}$, that we have previously employed. That difference, however, is one of phase-angle choices that amount to simple changes in the input and output reference planes on which the fields are defined. The new choices make the transformation symmetrical, which lends itself to greater insight into the quantum interference process.

We shall assume that the input modes are each in their single-photon state, so that their joint state is the product state $|\psi_{\text{in}}\rangle = |1\rangle_{S_{\text{in}}}|1\rangle_{I_{\text{in}}}$. What then is the joint state of the output modes? We know that it must be a pure state, because we are starting from a pure state and the beam splitter transformation is a unitary evolution. We know that it must contain exactly two photons, because the beam splitter transformation conserves energy and there are exactly two photons present at its input. Thus we can safely postulate

$$|\psi_{\text{out}}\rangle = c_{20}|2\rangle_{S_{\text{out}}}|0\rangle_{I_{\text{out}}} + c_{11}|1\rangle_{S_{\text{out}}}|1\rangle_{I_{\text{out}}} + c_{02}|0\rangle_{S_{\text{out}}}|2\rangle_{I_{\text{out}}}, \tag{2}$$

for the output state's number–state representation, where $|c_{20}|^2 + |c_{11}|^2 + |c_{02}|^2 = 1$. Furthermore, treating each input mode's input state as an independent, billiard-ball photon that is equally likely to be transmitted or reflected by the beam splitter, we could easily be led to conclude that

$$|c_{20}|^2 = |c_{02}|^2 = 1/4 \quad \text{and} \quad |c_{11}|^2 = 1/2, \tag{3}$$

so that

$$\Pr(N_{S_{\text{out}}} = n_S, N_{I_{\text{out}}} = n_I) = \begin{cases} 1/4, & \text{for } n_S = 2, n_I = 0 \\ 1/2, & \text{for } n_S = 1, n_I = 1 \\ 1/4, & \text{for } n_S = 0, n_I = 2 \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

for ideal (unity quantum efficiency) photon counting measurements on the output modes.

These results *seem* quite reasonable. There is only one way for both photons to emerge in the $\hat{a}_{S_{\text{out}}}$ mode: the $\hat{a}_{S_{\text{in}}}$ photon gets transmitted and the $\hat{a}_{I_{\text{in}}}$ photon gets reflected. Similarly, there is only one way for them to both emerge in the $\hat{a}_{I_{\text{out}}}$ mode: the $\hat{a}_{S_{\text{in}}}$ photon gets reflected and the $\hat{a}_{I_{\text{in}}}$ photon gets transmitted. On the other hand, there are *two* ways for one photon to emerge in each mode, i.e., both input photons are transmitted *or* both are reflected by the beam splitter. Because this billiard-ball picture says photon transmission and reflection is equally likely to occur at the 50-50 beam splitter, we get the photon counting distribution given above. Photons, however, are *not* billiard balls, as we know from our work on polarization entanglement. In the present context, their wave-like properties cause them to interfere at the 50-50 beam splitter, leading, as we will soon show, to the following output state

$$|\psi_{\text{out}}\rangle = \frac{|2\rangle_{S_{\text{out}}}|0\rangle_{I_{\text{out}}} + |0\rangle_{S_{\text{out}}}|2\rangle_{I_{\text{out}}}}{\sqrt{2}}. \tag{5}$$

Two things are worth noting before proceeding to the derivation: the input state was a product state, but the output state is entangled; and both photons *always* leave through the same output port. Why is it impossible to get one photon to appear in each output port? Quantum interference is the answer. In particular, we

must add the complex amplitudes for the two possible ways in which one photon can appear in each output port *before* taking the squared magnitude to calculate the photon counting probability for the event in which one photon is present at each output port. It is the nature of 50-50 beam splitting that the complex amplitudes for these two possibilities—both input photons transmitted or both reflected—have equal magnitudes but are π radians out of phase. Hence their complex amplitudes sum to zero, and we never get one photon emerging from each of the beam splitter's output ports.

To verify that the output state is as given in Eq. (5), let us assume that this equation is correct. The normally-ordered characteristic function for the output state then obeys,

$$\chi_N^{\text{out}}(\zeta_S^*, \zeta_I^*; \zeta_S, \zeta_I) \equiv \langle e^{\zeta_S \hat{a}_{S_{\text{out}}}^\dagger + \zeta_I \hat{a}_{I_{\text{out}}}^\dagger} e^{-\zeta_S^* \hat{a}_{S_{\text{out}}} - \zeta_I^* \hat{a}_{I_{\text{out}}}} \rangle = \tag{6}$$

$$\langle e^{\zeta_S \hat{a}_{S_{\text{out}}}^\dagger} e^{\zeta_I \hat{a}_{I_{\text{out}}}^\dagger} e^{-\zeta_S^* \hat{a}_{S_{\text{out}}}} e^{-\zeta_I^* \hat{a}_{I_{\text{out}}}} \rangle = \tag{7}$$

$$\frac{\left({}_{S_{\text{out}}}\langle 2| + \sqrt{2}\zeta_S {}_{S_{\text{out}}}\langle 1| + \frac{\zeta_S^2}{\sqrt{2}} {}_{S_{\text{out}}}\langle 0|\right) {}_{I_{\text{out}}}\langle 0| + \left({}_{I_{\text{out}}}\langle 2| + \sqrt{2}\zeta_I {}_{I_{\text{out}}}\langle 1| + \frac{\zeta_I^2}{\sqrt{2}} {}_{I_{\text{out}}}\langle 0|\right) {}_{S_{\text{out}}}\langle 0|}{\sqrt{2}} \times$$

$$\frac{\left(|2\rangle_{S_{\text{out}}} - \sqrt{2}\zeta_S^* |1\rangle_{S_{\text{out}}} + \frac{\zeta_S^{*2}}{\sqrt{2}} |0\rangle_{S_{\text{out}}}\right) |0\rangle_{I_{\text{out}}} + \left(|2\rangle_{I_{\text{out}}} - \sqrt{2}\zeta_I^* |1\rangle_{I_{\text{out}}} + \frac{\zeta_I^{*2}}{\sqrt{2}} |0\rangle_{I_{\text{out}}}\right) |0\rangle_{S_{\text{out}}}}{\sqrt{2}}$$

$$= 1 - |\zeta_S|^2 - |\zeta_I|^2 + |\zeta_S^2 + \zeta_I^2|^2/4, \tag{8}$$

where the second equality follows because $\hat{a}_{S_{\text{out}}}$ and $\hat{a}_{I_{\text{out}}}$ commute, and the third equality follows from series expansion of the exponentials plus the assumed output state. Now let us show that we can get this same result by starting from the *input* state and the beam splitter transformation.

From Eq. (1), we can easily show that

$$\chi_N^{\text{out}}(\zeta_S^*, \zeta_I^*; \zeta_S, \zeta_I) = \chi_N^{S_{\text{in}}}\left(\frac{j\zeta_S^* - \zeta_I^*}{\sqrt{2}}, \frac{-j\zeta_S - \zeta_I}{\sqrt{2}}\right) \chi_N^{I_{\text{in}}}\left(\frac{j\zeta_I^* - \zeta_S^*}{\sqrt{2}}, \frac{-j\zeta_I - \zeta_S}{\sqrt{2}}\right). \tag{9}$$

By series expansion of the exponentials in the characteristic functions on the right-hand side and the fact that the input modes are in their single-photon states, we then get

$$\chi_N^{\text{out}}(\zeta_S^*, \zeta_I^*; \zeta_S, \zeta_I) = (1 - |j\zeta_S + \zeta_I|^2/2)(1 - |j\zeta_I + \zeta_S|^2/2) \tag{10}$$

$$= 1 - |\zeta_S|^2 - |\zeta_I|^2 + |j\zeta_S + \zeta_I|^2 |j\zeta_I + \zeta_S|^2/4 \tag{11}$$

$$= 1 - |\zeta_S|^2 - |\zeta_I|^2 + |j\zeta_S + \zeta_I|^2 |\zeta_I - j\zeta_S|^2/4 \tag{12}$$

$$= 1 - |\zeta_S|^2 - |\zeta_I|^2 + |\zeta_S^2 + \zeta_I^2|^2/4, \tag{13}$$

which shows that Eq. (5) is indeed the output state.[2]

The calculational tools we have developed this semester allowed us to determine the output state for the slide 3 arrangement when its input modes are in their single-photon states. The work we have just completed, however, affords little insight into the quantum interference phenomenon that we described qualitatively before proceeding with our calculations. To gain that insight let us rewrite the input state in the following form,

$$|\psi_{\text{in}}\rangle = |1\rangle_{S_{\text{in}}} |1\rangle_{I_{\text{in}}} = \hat{a}_{S_{\text{in}}}^\dagger \hat{a}_{I_{\text{in}}}^\dagger |0\rangle_{S_{\text{in}}} |0\rangle_{I_{\text{in}}}, \tag{14}$$

where we have used photon-creating nature of $\hat{a}_{S_{\text{in}}}^\dagger$ and $\hat{a}_{I_{\text{in}}}^\dagger$. Now, because

$$\hat{a}_{S_{\text{in}}} = \frac{-j\hat{a}_{S_{\text{out}}} - \hat{a}_{I_{\text{out}}}}{\sqrt{2}} \quad \text{and} \quad \hat{a}_{I_{\text{in}}} = \frac{-\hat{a}_{S_{\text{out}}} - j\hat{a}_{I_{\text{out}}}}{\sqrt{2}} \tag{15}$$

is the inverse transformation associated with Eq. (1), and because the vacuum state $|0\rangle_{S_{\text{in}}} |0\rangle_{I_{\text{in}}}$ at the beam splitter's input leads to the vacuum state $|0\rangle_{S_{\text{out}}} |0\rangle_{I_{\text{out}}}$ at its output, Eq. (14) implies that

$$|\psi_{\text{out}}\rangle = \left(\frac{j\hat{a}_{S_{\text{out}}}^\dagger - \hat{a}_{I_{\text{out}}}^\dagger}{\sqrt{2}} \right) \left(\frac{-\hat{a}_{S_{\text{out}}}^\dagger + j\hat{a}_{I_{\text{out}}}^\dagger}{\sqrt{2}} \right) |0\rangle_{S_{\text{out}}} |0\rangle_{I_{\text{out}}} \tag{16}$$

$$= \frac{-j\hat{a}_{S_{\text{out}}}^{\dagger 2} - \hat{a}_{S_{\text{out}}}^\dagger \hat{a}_{I_{\text{out}}}^\dagger + \hat{a}_{I_{\text{out}}}^\dagger \hat{a}_{S_{\text{out}}}^\dagger - j\hat{a}_{I_{\text{out}}}^{\dagger 2}}{2} |0\rangle_{S_{\text{out}}} |0\rangle_{I_{\text{out}}} \tag{17}$$

$$= -j \frac{|2\rangle_{S_{\text{out}}} |0\rangle_{I_{\text{out}}} + |0\rangle_{S_{\text{out}}} |2\rangle_{I_{\text{out}}}}{\sqrt{2}}. \tag{18}$$

The four terms in the numerator on the right-hand side of the second equality represent the four possible ways in which the two photons that enter the beam splitter may emerge from that beam splitter. As promised, the second and third terms—which represent the events in which both are transmitted or both are reflected—have equal magnitudes and are π radians out of phase. As a result, these terms interfere *destructively*, and we never get one photon emerging from each of the beam splityer's output ports. Recall that the absolute phase of a ket is irrelevant, i.e., it does not affect quantum measurement statistics. Thus, this much quicker derivation does reproduce Eq. (5) while clearly revealing the underlying quantum intereference.

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[2]If you could not have guessed that Eq. (5) would be the output state, you could have converted $\chi_N^{\text{out}}(\zeta_S^*, \zeta_I^*; \zeta_S, \zeta_I)$ to the anti-normally ordered characteristic function. The operator-valued inverse Fourier transform of this anti-normally ordered characteristic function is the joint density operator for the output modes. By evaluating the number-ket matrix elements for this joint density operator you would have been led to conclude that the output modes were in the pure state given by Eq. (5).

Photon Indistinguishability

A key element in the quantum interference phenomenon that we just studied is photon indistinguishability, viz., two photons that are in the same mode cannot be distinguished from each other. As a prelude to our treatment of Hong-Ou-Mandel interferometry, let us re-examine the preceding quantum interference setup when the input photons are no longer indistinguishable. For this purpose, let us consider both the x and y polarizations of pure-tone, plane-wave pulses arriving at the beam splitter's input ports, i.e., $(\hat{a}_{S_x}^{\mathrm{in}}\vec{i}_x + \hat{a}_{S_y}^{\mathrm{in}}\vec{i}_y)e^{-j\omega_0 t}/\sqrt{T}$ and $(\hat{a}_{I_x}^{\mathrm{in}}\vec{i}_x + \hat{a}_{I_y}^{\mathrm{in}}\vec{i}_y)e^{-j\omega_0 t}/\sqrt{T}$ for $0 \le t \le T$. We will assume that the input state for these four modes is

$$|\psi_{\mathrm{in}}\rangle = |1\rangle_{S_{\mathrm{in}_x}}|0\rangle_{S_{\mathrm{in}_y}}(\cos(\theta)|1\rangle_{I_{\mathrm{in}_x}}|0\rangle_{I_{\mathrm{in}_y}} + \sin(\theta)|0\rangle_{I_{\mathrm{in}_x}}|1\rangle_{I_{\mathrm{in}_y}}), \qquad (19)$$

so that one photon enters each of the beam splitter's input ports, but they are polarized along $\vec{i}_x$ and $\vec{i}_\theta \equiv \cos(\theta)\vec{i}_x + \sin(\theta)\vec{i}_y$, respectively. For $\sin(\theta) \ne 0$, this makes the photons in the two input modes (at least partly) distinguishable. In particular, photon counting on the $\hat{a}_{S_y}^{\mathrm{in}}$ mode will never register a detection, but photon counting on the $\hat{a}_{I_y}^{\mathrm{in}}$ mode will register a detection with non-zero probability $\sin^2(\theta)$.

Our route to finding the output state for this situation will be a generalization of the simple quantum interference calculation that we gave at the end of the previous section.[3] The input state we have assumed can be written as follows,

$$|\psi_{\mathrm{in}}\rangle = \hat{a}_{S_x}^{\mathrm{in}\dagger}(\cos(\theta)\hat{a}_{I_x}^{\mathrm{in}\dagger} + \sin(\theta)\hat{a}_{I_y}^{\mathrm{in}\dagger})|0\rangle_{S_{\mathrm{in}_x}}|0\rangle_{S_{\mathrm{in}_y}}|0\rangle_{I_{\mathrm{in}_x}}|0\rangle_{I_{\mathrm{in}_y}} \qquad (20)$$

Using the fact that the vacuum state $|0\rangle_{S_{\mathrm{in}_x}}|0\rangle_{S_{\mathrm{in}_y}}|0\rangle_{I_{\mathrm{in}_x}}|0\rangle_{I_{\mathrm{in}_y}}$ at the beam splitter's input yields the vacuum state $|0\rangle_{S_{\mathrm{out}_x}}|0\rangle_{S_{\mathrm{out}_y}}|0\rangle_{I_{\mathrm{out}_x}}|0\rangle_{I_{\mathrm{out}_y}}$ at its output, and the beam-splitter relations

$$\hat{a}_{S_k}^{\mathrm{out}} = \frac{j\hat{a}_{S_k}^{\mathrm{in}} - \hat{a}_{I_k}^{\mathrm{in}}}{\sqrt{2}} \quad \text{and} \quad \hat{a}_{I_k}^{\mathrm{out}} = \frac{-\hat{a}_{S_k}^{\mathrm{in}} + j\hat{a}_{I_k}^{\mathrm{in}}}{\sqrt{2}}, \quad \text{for } k = x, y, \qquad (21)$$

we find that the output state is

$$
\begin{aligned}
|\psi_{\mathrm{out}}\rangle &= \left(\frac{j\hat{a}_{S_x}^{\mathrm{out}\dagger} - \hat{a}_{I_x}^{\mathrm{out}\dagger}}{\sqrt{2}}\right)\left[\cos(\theta)\left(\frac{-\hat{a}_{S_x}^{\mathrm{out}\dagger} + j\hat{a}_{I_x}^{\mathrm{out}\dagger}}{\sqrt{2}}\right) + \sin(\theta)\left(\frac{-\hat{a}_{S_y}^{\mathrm{out}\dagger} + j\hat{a}_{I_y}^{\mathrm{out}\dagger}}{\sqrt{2}}\right)\right] \\
&\quad \times |0\rangle_{S_{\mathrm{out}_x}}|0\rangle_{S_{\mathrm{out}_y}}|0\rangle_{I_{\mathrm{out}_x}}|0\rangle_{I_{\mathrm{out}_y}} \qquad (22) \\
&= -j\cos(\theta)\frac{|2\rangle_{S_{\mathrm{out}_x}}|0\rangle_{I_{\mathrm{out}_x}} + |0\rangle_{S_{\mathrm{out}_x}}|2\rangle_{I_{\mathrm{out}_x}}}{\sqrt{2}}|0\rangle_{S_{\mathrm{out}_y}}|0\rangle_{I_{\mathrm{out}_y}} \\
&\quad + \sin(\theta)\frac{-j|1\rangle_{S_{\mathrm{out}_x}}|1\rangle_{S_{\mathrm{out}_y}} - |1\rangle_{S_{\mathrm{out}_x}}|1\rangle_{I_{\mathrm{out}_y}} + |1\rangle_{I_{\mathrm{out}_x}}|1\rangle_{S_{\mathrm{out}_y}} - j|1\rangle_{I_{\mathrm{out}_x}}|1\rangle_{I_{\mathrm{out}_y}}}{2}.
\end{aligned}
$$
$$(23)$$

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[3]The characteristic function approach can also be employed, but it is considerably more tedious.

So, if we count the number of photons—summed over *both* polarizations—emerging from one of the output ports, there will be a probability $\sin^2(\theta)/2$ of getting one count. The difference in the polarization states of the incoming photons makes them at least partially distinguishable, and hence degrades the quantum interference that, for indistinguishable single-photon inputs, makes it impossible to observe only one count (with unity quantum efficiency detectors) at an output port. When $\theta = \pi/2$, the input photons are orthogonally polarized and thus completely distinguishable. In this case the output state is

$$|\psi_{\text{out}}\rangle = \frac{-j|1\rangle_{S_{\text{out}_x}}|1\rangle_{S_{\text{out}_y}} - |1\rangle_{S_{\text{out}_x}}|1\rangle_{I_{\text{out}_y}} + |1\rangle_{I_{\text{out}_x}}|1\rangle_{S_{\text{out}_y}} - j|1\rangle_{I_{\text{out}_x}}|1\rangle_{I_{\text{out}_y}}}{2}, \quad (24)$$

whose photon counting distribution,

$$\Pr(N_{S_x}^{\text{out}} + N_{S_y}^{\text{out}} = n_S, N_{I_x}^{\text{out}} + N_{I_y}^{\text{out}} = n_I) = \begin{cases} 1/4, & \text{for } n_S = 2, n_I = 0 \\ 1/2, & \text{for } n_S = 1, n_I = 1 \\ 1/4, & \text{for } n_S = 0, n_I = 2 \\ 0, & \text{otherwise,} \end{cases} \quad (25)$$

is in agreement with what is obtained from the simple billiard-ball photon picture given earlier.

Hong-Ou-Mandel Interferometry

Slide 4 shows a continuous-wave (cw) SPDC source driving a Hong-Ou-Mandel (HOM) interferometer. In the HOM configuration, two input beams are combined on a 50-50 beam splitter that can be moved to produce a differential time delay T in its input-output relation (see below). The output beams from the splitter are directed to a pair of photodetectors, each with quantum efficiency η but otherwise ideal, whose output photocurrents are processed by a coincidence counter. This counter measures the number of T_g-sec-long time intervals in which a *coincidence* has occurred, i.e., the number of T_g sec gate intervals in which *both* detectors have registered photodetections. Moreover, the coincidence measurement is performed as the differential delay T is varied. From our work on quantum interference, we expect that there will be no coincidences when a pair of indistinguishable photons—one in each input arm of the 50-50 beam splitter—enter the interferometer. Let's see if that is so for the photon pairs produced by cw SPDC.

We will assume that the SPDC source is a type-II system which is phase-matched at frequency degeneracy and operated in the low-gain regime. As shown on slide 4, a half-wave plate is used to rotate the signal-beam's polarization state so that it is co-polarized with the idler. In this case we can take the joint state of the y-polarized (non-vacuum state) beams that enter the HOM interferometer to be a zero-mean Gaussian that is characterized by its non-zero correlation functions, $K_{kk}^{(n)}(\tau) \equiv$

$\langle \hat{E}_k^\dagger(t+\tau)\hat{E}_k(t)\rangle$ for $k = S, I$, and $K_{SI}^{(p)}(\tau) \equiv \langle \hat{E}_S(t+\tau)\hat{E}_I(t)\rangle$. Using the results of Lecture 21 we have that these correlation functions are given by,

$$K_{SS}^{(n)}(\tau) = K_{II}^{(n)}(\tau) = \int \frac{d\omega}{2\pi} |\kappa|^2 l^2 \left(\frac{\sin(\omega\Delta k' l/2)}{\omega\Delta k' l/2}\right)^2 e^{j\omega\tau} \tag{26}$$

$$= \begin{cases} \dfrac{|\kappa|^2 l}{|\Delta k'|}\left(1 - \dfrac{|\tau|}{|\Delta k'| l}\right), & \text{for } |\tau| \leq |\Delta k'| l \\ 0, & \text{otherwise,} \end{cases} \tag{27}$$

and

$$K_{SI}^{(p)}(\tau) = \int \frac{d\omega}{2\pi} j\kappa l \frac{\sin(\omega\Delta k' l/2)}{\omega\Delta k' l/2} e^{j\omega(\tau + \Delta k' l/2)} = \begin{cases} \dfrac{j\kappa}{|\Delta k'|}, & \text{for } 0 \leq \tau \leq |\Delta k'| l \\ 0, & \text{otherwise,} \end{cases} \tag{28}$$

where we have assumed the $\Delta k' < 0$, as shown on the bottom of slide 4.

The field operators that illuminate the two photodetectors in the HOM setup will be taken to be

$$\hat{E}_{S_{\text{out}}}(t) = \frac{\hat{E}_S(t) + \hat{E}_I(t - T/2)}{\sqrt{2}} \quad \text{and} \quad \hat{E}_{I_{\text{out}}}(t) = \frac{-\hat{E}_S(t + T/2) + \hat{E}_I(t)}{\sqrt{2}}, \tag{29}$$

where T is the differential delay arising from the position of the beam splitter. Let $N_{S_{\text{out}}}$ and $N_{I_{\text{out}}}$ be the number of photons detected in the time interval $0 \leq t \leq T_g$ by the photodetectors that are illuminated by $\hat{E}_{S_{\text{out}}}(t)$ and $\hat{E}_{I_{\text{out}}}(t)$, respectively. From our work on quantum photodetection theory, we know that these classical random variables have statistics that are equivalent to those of the observables

$$\hat{N}_{S_{\text{out}}} \equiv \int_0^{T_g} dt\, \hat{E}'^\dagger_{S_{\text{out}}}(t)\hat{E}'_{S_{\text{out}}}(t) \quad \text{and} \quad \hat{N}_{I_{\text{out}}} \equiv \int_0^{T_g} dt\, \hat{E}'^\dagger_{I_{\text{out}}}(t)\hat{E}'_{I_{\text{out}}}(t), \tag{30}$$

where

$$\hat{E}'_{S_{\text{out}}}(t) \equiv \sqrt{\eta}\, \hat{E}_{S_{\text{out}}}(t) + \sqrt{1-\eta}\, \hat{E}_{\eta_S}(t) \text{ and } \hat{E}'_{I_{\text{out}}}(t) \equiv \sqrt{\eta}\, \hat{E}_{I_{\text{out}}}(t) + \sqrt{1-\eta}\, \hat{E}_{\eta_I}(t), \tag{31}$$

with $\hat{E}_{\eta_S}(t)$ and $\hat{E}_{\eta_I}(t)$ being in their vacuum states. It follow that

$$\langle \hat{N}_{S_{\text{out}}}\rangle = \left\langle \int_0^{T_g} dt\, \hat{E}'^\dagger_{S_{\text{out}}}(t)\hat{E}'_{S_{\text{out}}}(t)\right\rangle = \eta \left\langle \int_0^{T_g} dt\, \hat{E}^\dagger_{S_{\text{out}}}(t)\hat{E}_{S_{\text{out}}}(t)\right\rangle \tag{32}$$

$$= \frac{\eta}{2}\int_0^{T_g} dt\, [\langle \hat{E}_S^\dagger(t)\hat{E}_S(t)\rangle + \langle \hat{E}_I^\dagger(t-T/2)\hat{E}_I(t-T/2)\rangle] \tag{33}$$

$$= \frac{\eta}{2}\int_0^{T_g} dt\, [K_{SS}^{(n)}(0) + K_{II}^{(n)}(0)] = \frac{\eta|\kappa|^2 l T_g}{|\Delta k'|}. \tag{34}$$

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A similar calculation—which the reader should perform—will show that

$$\langle \hat{N}_{I_{\text{out}}} \rangle = \frac{\eta |\kappa|^2 l T_g}{|\Delta k'|}. \tag{35}$$

Signal and idler photons are produced in pairs by SPDC, so there should be no surprise that the preceding singles averages—the average number of counts in the gate interval for a single detector—should be identical. What we are interested in is the average number of coincidences in this gate interval. That will require a bit more work to determine.

In typical SPDC operation, the photon flux of the signal and idler, i.e., $K_{SS}^{(n)}(0) = K_{II}^{(n)}(0) = |\kappa|^2/|\Delta k'|$, will be less than (often much less than) $10^6 \, \text{s}^{-1}$. The duration of the coincidence gate, however, will usually be $T_g \sim 1 \, \text{ns}$. Consequently, we have

$$\langle \hat{N}_{S_{\text{out}}} \rangle = \langle \hat{N}_{I_{\text{out}}} \rangle \ll 1, \tag{36}$$

and so it is fair to say that each detector detects at most one photon during the time interval $0 \leq t \leq T_g$.[4] From this approximation we can use the classical photocount variables to identify coincidences by virtue of

$$N_{S_{\text{out}}} N_{I_{\text{out}}} = \begin{cases} 1, & \text{if there is a coincidence in } 0 \leq t \leq T_g \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

Invoking quantum photodetection theory again, we have that

$$C(T; T_g) \equiv \langle \hat{N}_{S_{\text{out}}} \hat{N}_{I_{\text{out}}} \rangle \tag{38}$$

gives the average number of coincidences in $0 \leq t \leq T_g$ as a function of the differential delay T and the gate duration T_g. To evaluate $C(T; T_g)$ we write

$$C(T; T_g) = \left\langle \int_0^{T_g} dt \, \hat{E}_{S_{\text{out}}}'^{\dagger}(t) \hat{E}_{S_{\text{out}}}'(t) \int_0^{T_g} du \, \hat{E}_{I_{\text{out}}}'^{\dagger}(u) \hat{E}_{I_{\text{out}}}'(u) \right\rangle, \tag{39}$$

combine the product of integrals into a double integral and use the fact that $\hat{E}_{S_{\text{out}}}(t)$ and $\hat{E}_{I_{\text{out}}}(u)$ commute with each other and with each other's adjoint, obtaining

$$C(T; T_g) = \int_0^{T_g} dt \int_0^{T_g} du \, \langle \hat{E}_{S_{\text{out}}}'^{\dagger}(t) \hat{E}_{I_{\text{out}}}'^{\dagger}(u) \hat{E}_{S_{\text{out}}}'(t) \hat{E}_{I_{\text{out}}}'(u) \rangle \tag{40}$$

$$= \eta^2 \int_0^{T_g} dt \int_0^{T_g} du \, \langle \hat{E}_{S_{\text{out}}}^{\dagger}(t) \hat{E}_{I_{\text{out}}}^{\dagger}(u) \hat{E}_{S_{\text{out}}}(t) \hat{E}_{I_{\text{out}}}(u) \rangle. \tag{41}$$

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[4]A rigorous proof of this statement requires a bit more work than indicated here, but will be omitted.

Because $\hat{E}_S(t)$ and $\hat{E}_I(t)$ are in a jointly Gaussian state, and $\hat{E}_{S_\text{out}}(t)$ and $\hat{E}_{I_\text{out}}(t)$ are obtained from a linear transformation of these field operators, they too are in a jointly Gaussian state. Thus the quantum version of the Gaussian moment factoring theorem allows us to reduce the fourth-order field-operator moment in $C(T; T_g)$ to sums of products of second-order field-operator moments, viz.,[5]

$$
\begin{aligned}
C(T; T_g) &= \eta^2 \int_0^{T_g} dt \int_0^{T_g} du \, \langle \hat{E}_{S_\text{out}}^\dagger(t) \hat{E}_{S_\text{out}}(t) \rangle \langle \hat{E}_{I_\text{out}}^\dagger(u) \hat{E}_{I_\text{out}}(u) \rangle \\[2mm]
&+ \eta^2 \int_0^{T_g} dt \int_0^{T_g} du \, |\langle \hat{E}_{S_\text{out}}(t) \hat{E}_{I_\text{out}}(u) \rangle|^2 \quad\quad (42) \\[2mm]
&= \langle \hat{N}_{S_\text{out}} \rangle \langle \hat{N}_{I_\text{out}} \rangle + \eta^2 \int_0^{T_g} dt \int_0^{T_g} du \, |K^{(p)}_{S_\text{out} I_\text{out}}(t-u)|^2 \quad\quad (43) \\[2mm]
&= \left(\frac{\eta |\kappa|^2 l T_g}{|\Delta k'|} \right)^2 + \eta^2 T_g \int_{-T_g}^{T_g} d\tau \, |K^{(p)}_{S_\text{out} I_\text{out}}(\tau)|^2 \left(1 - \frac{|\tau|}{T_g} \right) \quad\quad (44) \\[2mm]
&\approx \left(\frac{\eta |\kappa|^2 l T_g}{|\Delta k'|} \right)^2 + \eta^2 T_g \int_{-\infty}^{\infty} d\tau \, |K^{(p)}_{S_\text{out} I_\text{out}}(\tau)|^2, \quad\quad (45)
\end{aligned}
$$

where the approximation assumes that $T_g \gg |T| + |\Delta k'| l$. In typical SPDC-HOM experiments, T and $|\Delta k'| l$ are on the order of psec, so this approximation is well justified for $T_g \sim 1\,\text{ns}$.

To go further we use the beam splitter relation for $\hat{E}_{S_\text{out}}(t)$ and $\hat{E}_{I_\text{out}}(t)$ to express their phase-sensitive cross-correlation function in terms of the corresponding cross-correlation function for $\hat{E}_S(t)$ and $\hat{E}_I(t)$, i.e.,

$$
\begin{aligned}
K^{(p)}_{S_\text{out} I_\text{out}}(\tau) &= \left\langle \frac{\hat{E}_S(t+\tau) + \hat{E}_I(t+\tau-T/2)}{\sqrt{2}} \frac{-\hat{E}_S(t+T/2) + \hat{E}_I(t)}{\sqrt{2}} \right\rangle \quad (46) \\[2mm]
&= \frac{K^{(p)}_{SI}(\tau) - K^{(p)}_{SI}(-\tau + T)}{2}. \quad\quad (47)
\end{aligned}
$$

---

[5]See the random processes notes for a brief discussion of the Gaussian moment factoring theorem for real-valued classical random variables. The corresponding result for the quantum case that we will use several times today is as follows. If $\hat{E}_a(t)$ and $\hat{E}_b(t)$ are in a zero-mean jointly Gaussian state then

$$
\begin{aligned}
\langle \hat{E}_a^\dagger(t) \hat{E}_b^\dagger(u) \hat{E}_a(t) \hat{E}_b(u) \rangle &= \langle \hat{E}_a^\dagger(t) \hat{E}_a(t) \rangle \langle \hat{E}_b^\dagger(u) \hat{E}_b(u) \rangle + \langle \hat{E}_a^\dagger(t) \hat{E}_b(u) \rangle \langle \hat{E}_b^\dagger(u) \hat{E}_a(t) \rangle \\[2mm]
&+ \langle \hat{E}_a^\dagger(t) \hat{E}_b^\dagger(u) \rangle \langle \hat{E}_a(t) \hat{E}_b(u) \rangle.
\end{aligned}
$$

This same result can be used for a single field, $\hat{E}(t)$, that is in a zero-mean Gaussian state by setting $\hat{E}_a(s) = \hat{E}_b(s) = \hat{E}(s)$ in the preceding expression for $s = t, u$

whence

$$C(T; T_g) \approx \left(\frac{\eta |\kappa|^2 l T_g}{|\Delta k'|} \right)^2 + \frac{\eta^2 T_g}{4} \int_{-\infty}^{\infty} d\tau \, |K_{SI}^{(p)}(\tau) - K_{SI}^{(p)}(-\tau + T)|^2. \qquad (48)$$

For $|T| \geq 2|\Delta k'|l$, the two terms inside the integrand's magnitude squared are non-overlapping time functions, so that

$$C(T; T_g) \approx \left(\frac{\eta |\kappa|^2 l T_g}{|\Delta k'|} \right)^2 + \frac{\eta^2 T_g}{2} \int_{-\infty}^{\infty} d\tau \, |K_{SI}^{(p)}(\tau)|^2 \qquad (49)$$

$$= \left(\frac{\eta |\kappa|^2 l T_g}{|\Delta k'|} \right)^2 + \frac{\eta^2 |\kappa|^2 l T_g}{2|\Delta k'|}. \qquad (50)$$

In the low-flux limit that we have assumed, wherein $|\kappa|^2 l/|\Delta k'| \ll 1$, the first term on the right in Eq. (50) can be neglected in comparison with the second. Experimentalists refer to the first term as the "accidental" coincidences, not the "true" coincidences that are counted by the second term . Thus we shall suppress these accidentals and say that

$$C(T; T_g) \approx \frac{\eta^2 T_g}{4} \int_{-\infty}^{\infty} d\tau \, |K_{SI}^{(p)}(\tau) - K_{SI}^{(p)}(-\tau + T)|^2, \qquad (51)$$

gives the HOM interferometer's coincidence count in the T_g-sec gate interval. As shown on slide 5, $C(T; T_g)$ drops to zero when $T = |\Delta k'|l$, despite the average number of singles on each detector being unaffected by the differential delay T. This $C(T; T_g) \to 0$ as $T \to |\Delta k'|l$ behavior is called the HOM dip, and it occurs because for this value of the differential delay we get $K_{SI}^{(p)}(\tau) = K_{SI}^{(p)}(-\tau + T)$. The HOM dip is the signature of quantum interference between indistinguishable photons that we saw earlier in this lecture, as we will now explain.

In type-II phase-matched, frequency-degenerate, cw SPDC, a single pump photon can spontaneously fission into a signal-idler photon pair at some z-plane within the $\chi^{(2)}$ crystal. Inside the crystal, the signal and idler photons propagate at their respective group velocities,

$$v_{gS} = \left(\frac{dk_S(\omega_P/2 + \omega)}{d\omega} \bigg|_{\omega=0} \right)^{-1} \quad \text{and} \quad v_{gI} = \left(-\frac{dk_I(\omega_P/2 - \omega)}{d\omega} \bigg|_{\omega=0} \right)^{-1}. \qquad (52)$$

Thus the component photons of this signal-idler pair separate as they propagate from where they were created to the crystal's exit facet at $z = l$, because

$$\Delta k' = -\frac{dk_S(\omega_P/2 + \omega)}{d\omega} \bigg|_{\omega=0} - \frac{dk_I(\omega_P/2 - \omega)}{d\omega} \bigg|_{\omega=0} = -\frac{1}{v_{gS}} + \frac{1}{v_{gI}} \neq 0. \qquad (53)$$

We have assumed $\Delta k' < 0$, which means that inside the nonlinear crystal the signal propagates *slower* than the idler does. So, consistent with the $K_{SI}^{(p)}(\tau) = \langle \hat{E}_S(t +$

$\tau)\hat{E}_I(t)\rangle$ sketch on slide 4, a signal-idler pair created at $z = l$ is correlated at $\tau = 0$, while a signal-idler pair created at $z = 0$ will be correlated at $\tau = |\Delta k'|l$, and signal-idler pairs created at intermediate z-planes within the $\chi^{(2)}$ crystal will be correlated at τ values intermediate between these two extremes.[6] In order to make the two component photons of a signal-idler pair be indistinguishable—so that quantum interference will occur in the HOM interferometer—we need the differential delay T to compensate for the idler's group velocity advantage in the crystal. That is indeed is what $T = |\Delta k'|l$ accomplishes.

The Biphoton and Generation of Polarization Entanglement

Our Gaussian-state analysis of the HOM dip obtained with SPDC, although rigorous, is much more elaborate than what almost all experimentalists—and for that matter almost all theorists—ordinarily employ. Instead, for the low-gain, low-flux regime in which we evaluated the HOM dip, they would say that the output state from frequency-degenerate, cw SPDC is

$$|\psi_{SI}\rangle = |\mathbf{0}\rangle_S|\mathbf{0}\rangle_I + \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\, j\kappa l\frac{\sin(\omega\Delta k'l/2)}{\omega\Delta k'l/2}e^{j\omega\Delta k'l/2}|\omega_P/2 + \omega\rangle_S|\omega_P/2 - \omega\rangle_I. \quad (54)$$

Here: $|\mathbf{0}\rangle_S$ and $|\mathbf{0}\rangle_I$ are the vacuum states of $\hat{E}_S(t)$ and $\hat{E}_I(t)$, respectively; $|\omega_P/2+\omega\rangle_S$ is the state in which $\hat{E}_S(t)e^{-j\omega_P t/2}$ has one photon at frequency $\omega_P/2 + \omega$ and no photons at other frequencies; and $|\omega_P/2 - \omega\rangle_I$ is the state in which $\hat{E}_I(t)e^{-j\omega_P t/2}$ has one photon at frequency $\omega_P/2 - \omega$ and no photons at other frequencies. There are many things worth noting about Eq. (54).

- The state $|\psi_{SI}\rangle$ is *not* properly normalized.

- Because we are in the low-gain, low-flux regime for SPDC, the vacuum term in $|\psi_{SI}\rangle$ dominates its non-vacuum term.

- The non-vacuum term is called the biphoton state. It is an entangled state in which a signal photon at frequency $\omega_P/2+\omega$ is accompanied by an idler photon at frequency $\omega_P/2 - \omega$, in accord with energy conservation for pump-photon fission.

- In coincidence-counting experiments we can *post-select* for the biphoton state by including in our data processing only those measurements in which both a signal photon and an idler photon were detected.

- Equation (54) reproduces the rigorous Gaussian-state results for the first and second moments of $\hat{E}_S(t)$ and $\hat{E}_I(t)$. Thus, in the low-gain, low-flux regime for

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[6]Remember, that $K_{SI}^{(p)}(\tau)$ is the phase-sensitive cross-correlation function *outside* the nonlinear crystal, where both the signal and idler photons propagate at the vacuum light speed, c.

cw SPDC—in which at most one signal-idler pair is observed over the measurement interval—it is appropriate to use Eq. (54) in lieu of the rigorous jointly Gaussian state of the signal and idler beams.[7]

To exercise what we have just said about the biphoton state, let us use that approach to characterize the scheme shown on slide 6 for generating polarization-entangled photon pairs from SPDC. Here we have two type-II phase-matched, cw SPDC sources pumped in antiphased manner. They need not be operated at frequency degeneracy, i.e., the center frequencies for the signal and idler may be $\omega_S \neq \omega_I$ so long as $\omega_S + \omega_I = \omega_P$ and $k_S(\omega_S) + k_I(\omega_I) = k_P(\omega_P)$. As shown on slide 6, we have oriented the nonlinear crystals for these two sources such that a polarizing beam splitter is able to direct both signal beams to one of its output ports and both idler beams to its other output port. The joint state of the two SPDC sources, from Eq. (54), is

$$
\begin{aligned}
|\psi_{\text{in}}\rangle &= \left(|\mathbf{0}\rangle_{S_x}|\mathbf{0}\rangle_{I_y} + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} j\kappa l \frac{\sin(\omega \Delta k' l/2)}{\omega \Delta k' l/2} e^{j\omega \Delta k' l/2} |\omega_S + \omega\rangle_{S_x}|\omega_I - \omega\rangle_{I_y}\right) \\
&\otimes \left(|\mathbf{0}\rangle_{S_y}|\mathbf{0}\rangle_{I_x} - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} j\kappa l \frac{\sin(\omega \Delta k' l/2)}{\omega \Delta k' l/2} e^{j\omega \Delta k' l/2} |\omega_S + \omega\rangle_{S_y}|\omega_I - \omega\rangle_{I_x}\right).
\end{aligned}
$$

(55)

On the right-hand side of the first line we have only included the state of the x-polarized signal and the y-polarized idler, and on the second line we have only included the state of the y-polarized signal and the x-polarized idler, as the other polarizations are all in their vacuum states. The minus sign in the second term on the right-hand side of the second line is due to the antiphased pumping.[8]

After the polarizing beam splitter, and using the fact that the vacuum terms predominate on both lines of Eq. (55), we get the following output state to first order of smallness:

$$
|\psi_{\text{out}}\rangle = |\mathbf{0}\rangle_{S_x}|\mathbf{0}\rangle_{S_y}|\mathbf{0}\rangle_{I_x}|\mathbf{0}\rangle_{I_y} +
$$

$$
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} j\kappa l \frac{\sin(\omega \Delta k' l/2)}{\omega \Delta k' l/2} e^{j\omega \Delta k' l/2} (|\omega_S + \omega\rangle_{S_x}|\omega_I - \omega\rangle_{I_y} - |\omega_S + \omega\rangle_{S_y}|\omega_I - \omega\rangle_{I_x}).
$$

(56)

Equation (56) is a frequency-entangled, polarization-singlet state. Let us exhibit its polarization entanglement by returning to the full Gaussian-state characterization for the output state from the slide 6 system.

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[7]The interested reader may want to try to analyze the HOM dip using Eq. (54) as the signal-idler input state.

[8]Recall that κ is proportional to the complex amplitude of the pump field. So with κ defined for one SPDC source, antiphased pumping flips the sign of κ for the other SPDC source.

The signal and idler beams at the output ports of the polarizing beam splitter on slide 6 are in a zero-mean jointly Gaussian state with the following non-zero correlation functions:

$$K_{S_x S_x}^{(n)}(\tau) = K_{S_y S_y}^{(n)}(\tau) = K_{I_x I_x}^{(n)}(\tau) = K_{I_y I_y}^{(n)}(\tau) \tag{57}$$

$$= K^{(n)}(\tau) \equiv \begin{cases} \dfrac{|\kappa|^2 l}{|\Delta k'|}\left(1 - \dfrac{|\tau|}{|\Delta k'|l}\right), & \text{for } |\tau| \leq |\Delta k'|l \\ 0, & \text{otherwise,} \end{cases} \tag{58}$$

and

$$K_{S_x I_y}^{(p)}(\tau) = -K_{S_y I_x}^{(p)}(\tau) = K^{(p)}(\tau) \equiv \begin{cases} \dfrac{j\kappa}{|\Delta k'|}, & \text{for } 0 \leq \tau \leq |\Delta k'|l \\ 0, & \text{otherwise,} \end{cases} \tag{59}$$

where we have continued to assume that $\Delta k' < 0$.

Suppose that we perform the following coincidence counting experiment. We use a polarization analysis system to illuminate one photodetector with the $\mathbf{i} \equiv \alpha \vec{i}_x + \beta \vec{i}_y$ component of the signal beam and we use another polarization analysis system to illuminate a second photodetector with the orthogonal, $\mathbf{i'} \equiv \beta^* \vec{i}_x - \alpha^* \vec{i}_y$, polarization of the idler beam, where $|\alpha|^2 + |\beta|^2 = 1$. Paralleling what we did for the HOM interferometer, we can show that the average number of coincidence counts for this setup satisfies

$$C_{S_{\mathbf{i}} I_{\mathbf{i'}}}(T; T_g) = \eta^2 T_g \int_{-\infty}^{\infty} d\tau\, |K_{S_{\mathbf{i}} I_{\mathbf{i'}}}^{(p)}(\tau)|^2, \tag{60}$$

where

$$K_{S_{\mathbf{i}} I_{\mathbf{i'}}}^{(p)}(\tau) = \langle [\alpha^* \hat{E}_{S_x}(t+\tau) + \beta^* \hat{E}_{S_y}(t+\tau)][\beta \hat{E}_{I_x}(t) - \alpha \hat{E}_{I_y}(t)] \rangle \tag{61}$$

$$= -|\alpha|^2 K_{S_x I_y}^{(p)}(\tau) + |\beta|^2 K_{S_y I_x}^{(p)}(\tau) = -K^{(p)}(\tau), \tag{62}$$

so that

$$C_{S_{\mathbf{i}} I_{\mathbf{i'}}}(T; T_g) = \frac{\eta^2 |\kappa|^2 l T_g}{|\Delta k'|}. \tag{63}$$

Similar calculations—which the reader should attempt—yield

$$C_{S_{\mathbf{i'}} I_{\mathbf{i}}}(T; T_g) = \frac{\eta^2 |\kappa|^2 l T_g}{|\Delta k'|} \quad \text{and} \quad C_{S_{\mathbf{i}} I_{\mathbf{i}}}(T; T_g) = C_{S_{\mathbf{i'}} I_{\mathbf{i'}}}(T; T_g) = 0. \tag{64}$$

These are the continuous-time signatures of singlet-state polarization entanglement, cf. the results from Lecture 13 for the a pair of antiphased two-mode parametric amplifiers. In particular, when the signal and idler outputs from the polarizing beam splitter on slide 6 undergo polarization analysis in an arbitrary common basis, whenever a coincidence occurs, the signal and idler detections will have occurred in orthogonal polarizations.

Photon Twins from an Optical Parametric Amplifier

Our last task for today will be to develop the continuous-time version of the photon-twins signature of nonclassical light that we developed earlier this semester within the two-mode construct. In effect, we have already seen that signature in the biphoton state, i.e., SPDC produces signal and idler photons in pairs via photon fission. However, instead of working with the SPDC source, we shall employ the lumped-element optical parametric amplifier model from Lecture 21 to study photon twinning. SPDC produces signal-idler pairs at rates $\leq 10^6 \, \text{s}^{-1}$ in typical systems. This pair rate is well resolved by the $\sim 1 \, \text{ns}$ response times of fast photodetectors. A doubly-resonant OPA that is capable of producing $10 \, \text{dB}$ of quadrature-noise squeezing in a $10 \, \text{MHz}$ bandwidth will produce $\sim 10^8$ signal-idler pairs per second in that bandwidth. Although single signal-idler pairs from an OPA can still be resolved by a fast photodetector, this source is much closer to the system that experimentalists have actually used for photon-twins experiments, viz., the optical parametric oscillator (OPO). The OPO is an OPA pumped above its oscillation threshold. Its signal and idler outputs can easily reach rates of 10^{13} photons per second. However, rather than introduce yet another $\chi^{(2)}$ analysis into the mix, we shall content ourselves with demonstrating the photon twins behavior of the OPA.

The setup of interest is shown on slide 10. The signal and idler outputs from a doubly-resonant, type-II phase-matched OPA are separated—with a polarizing beam splitter—and directed to ideal (unity quantum efficiency) photodetectors. The resulting photocounts, N_S and N_I, for the time interval $0 \leq t \leq T$ are then combined to yield the photocount difference $\Delta N \equiv N_S - N_I$. In semiclassical photodetection, N_S and N_I are Poisson distributed, given the powers that illuminate their respective photodetectors during $0 \leq t \leq T$. Thus, because the shot noises from different detectors are statistically independent, we know that their difference, ΔN, will have a variance that satisfies the following shot-noise limit:

$$\text{var}(\Delta N) \equiv \langle [\Delta N - \langle \Delta N \rangle]^2 \rangle \geq \langle N_S \rangle + \langle N_I \rangle. \tag{65}$$

From quantum theory, however, we expect the variance of ΔN to be zero, because signal and idler photons are created in pairs, and we are using ideal photodetectors, i.e., ideal photon counters. To probe whether this is really so, we start from the quantum observables, $\hat{N}_S, \hat{N}_I$, and $\widehat{\Delta N} \equiv \hat{N}_S - \hat{N}_I$, whose measurement statistics coincide with those of the classical random variables N_S, N_I, and ΔN, where

$$\hat{N}_S = \int_0^T dt \, \hat{E}_S^{\text{out}\dagger}(t) \hat{E}_S^{\text{out}}(t) \quad \text{and} \quad \hat{N}_I = \int_0^T dt \, \hat{E}_I^{\text{out}\dagger}(t) \hat{E}_I^{\text{out}}(t), \tag{66}$$

with $\{\, \hat{E}_m^{\text{out}}(t) : m = S, I \,\}$ being the baseband field operators for the OPA's outputs. The mean and variance calculations that we must perform are similar to, but simpler than, what we have already done for HOM interferometry.

In Lecture 21 we established that the OPA's signal and idler outputs were in a zero-mean, stationary, jointly Gaussian state whose non-zero correlation functions are,

$$K_{mm}^{(n)}(\tau) = \langle \hat{E}_m^{\text{out}\dagger}(t)\hat{E}_m^{\text{out}}(u)\rangle = \frac{G\Gamma}{2}\left[\frac{e^{-(1-G)\Gamma|\tau|}}{1-G} - \frac{e^{-(1+G)\Gamma|\tau|}}{1+G}\right], \text{ for } m = S, I, \quad (67)$$

$$K_{SI}^{(p)}(\tau) = \frac{G\Gamma}{2}\left[\frac{e^{-(1-G)\Gamma|\tau|}}{1-G} + \frac{e^{-(1+G)\Gamma|\tau|}}{1+G}\right]. \quad (68)$$

Thus, the average signal and idler photocounts obey

$$\langle \hat{N}_m\rangle = \int_0^T \mathrm{d}t\, \langle \hat{E}_m^{\text{out}\dagger}(t)\hat{E}_m^{\text{out}}(t)\rangle = \int_0^T \mathrm{d}t\, K_{mm}^{(n)}(0) = \frac{G^2\Gamma T}{1-G^2}, \quad \text{for } m = S, I. \quad (69)$$

Equation (69) is what we expect: signal and idler photons are created in pairs so the average number of signal and idler counts in any T-sec-long interval should coincide. Equation (69) implies,

$$\langle \widehat{\Delta N}\rangle = \langle \hat{N}_S\rangle - \langle \hat{N}_I\rangle = 0, \quad (70)$$

so that the nonclassical photon-twins signature we are seeking becomes

$$\langle (\widehat{\Delta N})^2\rangle < \langle \hat{N}_S\rangle + \langle \hat{N}_I\rangle = \frac{2G^2\Gamma T}{1-G^2}. \quad (71)$$

Evaluating $\langle (\widehat{\Delta N})^2\rangle$ mimics, in several respects, what we did to find the average coincidence count for HOM interferometry. We start from

$$\langle (\widehat{\Delta N})^2\rangle = \left\langle \int_0^T \mathrm{d}t\, [\hat{E}_S^{\text{out}\dagger}(t)\hat{E}_S^{\text{out}}(t) - \hat{E}_I^{\text{out}\dagger}(t)\hat{E}_I^{\text{out}}(t)]\right.$$

$$\left. \times \int_0^T \mathrm{d}u\, [\hat{E}_S^{\text{out}\dagger}(u)\hat{E}_S^{\text{out}}(u) - \hat{E}_I^{\text{out}\dagger}(u)\hat{E}_I^{\text{out}}(u)]\right\rangle. \quad (72)$$

Then, we combine the product of single integrals into a double integral and use the fact that $\hat{E}_S^{\text{out}}(t)$ and $\hat{E}_I^{\text{out}}(t)$ commute with each other and with each other's adjoint but have the δ-function commutator with their own adjoints. This leads us to

$$\langle (\widehat{\Delta N})^2\rangle = \int_0^T \mathrm{d}t\, \langle \hat{E}_S^{\text{out}\dagger}(t)\hat{E}_S^{\text{out}}(t)\rangle + \int_0^T \mathrm{d}u\, \langle \hat{E}_I^{\text{out}\dagger}(u)\hat{E}_I^{\text{out}}(u)\rangle$$

$$+ \int_0^T \mathrm{d}t \int_0^T \mathrm{d}u\, \langle \hat{E}_S^{\text{out}\dagger}(t)\hat{E}_S^{\text{out}\dagger}(u)\hat{E}_S^{\text{out}}(t)\hat{E}_S^{\text{out}}(u)\rangle$$

$$+ \int_0^T \mathrm{d}t \int_0^T \mathrm{d}u\, \langle \hat{E}_I^{\text{out}\dagger}(t)\hat{E}_I^{\text{out}\dagger}(u)\hat{E}_I^{\text{out}}(t)\hat{E}_I^{\text{out}}(u)\rangle$$

$$- 2\int_0^T \mathrm{d}t \int_0^T \mathrm{d}u\, \langle \hat{E}_S^{\text{out}\dagger}(t)\hat{E}_I^{\text{out}\dagger}(u)\hat{E}_S^{\text{out}}(t)\hat{E}_I^{\text{out}}(u)\rangle, \quad (73)$$

which can be reduced—by means of the quantum form of Gaussian moment factoring and the specific forms we have for $K_{SS}^{(n)}(\tau), K_{SS}^{(n)}(\tau)$, and $K_{SI}^{(p)}(\tau)$—to

$$\langle(\widehat{\Delta N})^2\rangle = \frac{2G^2\Gamma T}{1-G^2} - \frac{2G^2\Gamma^2 T}{1-G^2}\int_{-T}^{T}\mathrm{d}\tau\left(1 - \frac{|\tau|}{T}\right)e^{-2\Gamma|\tau|} \tag{74}$$

$$= \frac{2G^2\Gamma T}{1-G^2}\frac{1-e^{-2\Gamma T}}{2\Gamma T}. \tag{75}$$

Equation (75) falls below the semiclassical (shot-noise) limit of $2G^2\Gamma T/(1-G^2)$ for all $T > 0$, but only equals zero in the limit of $T \to \infty$. Thus, even though we have perfectly efficient detectors, the number of signal photons counted *exactly* matches the number of idler photon counted *only* in the infinite integration-time limit. It is easy to understand why that should be so. Although signal and idler photons are created in pairs, within the OPA, each photon from any pair may stay inside the doubly-resonant cavity for many cavity lifetimes, i.e., many times the reciprocal bandwidth $1/\Gamma$. Only when we have counted photons for many cavity lifetimes are we assured that both photons from almost every pair have exited the cavity. Hence it is only in this limit that we are guaranteed to get a photocount difference whose variance is well below the semiclassical limit. Indeed, Eq. (75) shows that the normalized variance of the photocount difference satisfies

$$\frac{\langle(\widehat{\Delta N})^2\rangle}{\langle\hat{N}_S\rangle + \langle\hat{N}_I\rangle} \approx \begin{cases} 1, & \text{for } \Gamma T \ll 1 \\ 1/2\Gamma T \ll 1, & \text{for } \Gamma T \gg 1. \end{cases} \tag{76}$$

We have plotted this normalized variance in the right panel on slide 11—along with the corresponding semiclassical value of unity—as a function of ΓT. Also shown on that slide is a plot of the normalized variance for the individual photon counts,[9]

$$\frac{\langle[\Delta\hat{N}_S]^2\rangle}{\langle\hat{N}_S\rangle} = \frac{\langle[\Delta\hat{N}_I]^2\rangle}{\langle\hat{N}_I\rangle}, \tag{77}$$

versus ΓT for $G^2 = 0.01$. These individual variances are super-Poissonian, i.e., they exceed the shot-noise limit. Moreover, they are the *same* as would be found from the semiclassical theory of photodetection. That agreement between the semiclassical and quantum theories for the individual photocount variances is no accident. As in the case of the two-mode parametric amplifier, the reduced density operators for the signal and idler beams from our doubly-resonant OPA are classical states, i.e., they have proper P representations.[10]

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[9]We leave the derivation of these photocount variances as an exercise for the reader. Their derivation is similar to what we have done for $\langle(\widehat{\Delta N})^2\rangle$.

[10]A zero-mean Gaussian state that has no phase-sensitive correlation can be shown to be a classical state, but we shall not provide the proof.

The Road Ahead

In the next lecture, we shall survey a collection of additional applications of non-classical light: binary optical communication with squeezed states; phase-sensing interferometry with squeezed states; super-dense coding with entangled states; and quantum lithography with "N00N" states.

6.453 Quantum Optical Communication

Fall 2016