Lecture 11: Solutions to Laplace's Equation In Polar and Spherical Coordinates

I. Solution's to Laplace's Equation in Polar Coordinates $\left(\dfrac{\partial}{\partial z} = 0 \right)$

    A. Product Solution

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \cancel{\frac{\partial^2 \phi}{\partial z^2}}^{\,0} = 0$$
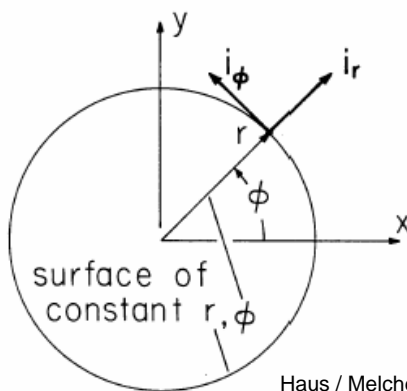
$$\Phi(r, \phi) = R(r) F(\phi)$$

$$\frac{F(\phi)}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{R(r)}{r^2} \frac{d^2 F}{d\phi^2} = 0 \qquad \left| \begin{array}{l} \text{Multiply through} \\[4pt] \text{by} \quad \dfrac{r^2}{R(r) F(\phi)} \end{array} \right.$$

$$\underbrace{\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right)}_{m^2} + \underbrace{\frac{1}{F} \frac{d^2 F}{d\phi^2}}_{-m^2} = 0$$

$$\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = m^2 \quad \Rightarrow \quad r \frac{d}{dr} \left(r \frac{dR}{dr} \right) - m^2 R = 0$$

$$\frac{1}{F} \frac{d^2 F}{d\phi^2} = -m^2 \quad \Rightarrow \quad \frac{d^2 F}{d\phi^2} + m^2 F = 0$$

Haus / Melcher **Figure 5.7.1** Polar coordinate system.
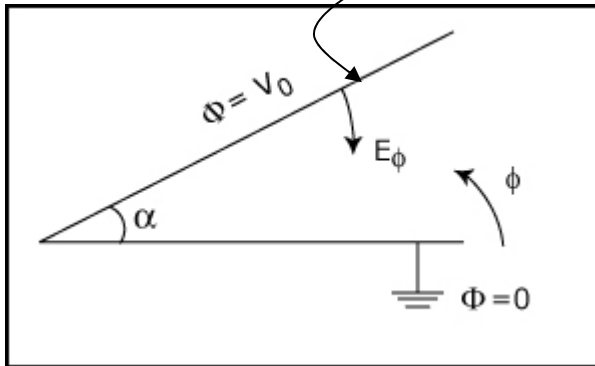
Courtesy of MIT Press. Used with permission.

B. m=0 Solutions (Zero Separation Constant Solutions)

$$r \frac{dR}{dr} = C \implies R = C \ln r + D$$

$$\frac{d^2 F}{d\phi^2} = 0 \implies F = A\phi + B$$

Potential of line charge

$$\Phi(r, \phi) = R(r) F(\phi) = A_1 + A_2 \phi + A_3 \ln r + A_4 \phi \ln r$$

$$\Phi(\phi = 0) = 0 \\ \Phi(\phi = \alpha) = V_0 \Bigg\} \implies \Phi(\phi) = \frac{V_0 \phi}{\alpha} \implies \bar{E} = -\nabla \Phi = - \left[\bar{i}_r \frac{\partial \Phi}{\partial r} + \bar{i}_\phi \frac{1}{r} \frac{\partial \Phi}{\partial \phi} + \bar{i}_z \frac{\partial \Phi}{\partial z} \right]$$

$$E_\phi = - \frac{1}{r} \frac{\partial \Phi}{\partial \phi} = - \frac{V_0}{\alpha r}$$

$$\sigma_s (r, \phi = 0) = \varepsilon_0 E_\phi (r, \phi = 0) = - \frac{\varepsilon_0 V_0}{\alpha r}$$

$$\sigma_s (r, \phi = \alpha) = - \varepsilon_0 E_\phi (r, \phi = \alpha) = + \frac{\varepsilon_0 V_0}{\alpha r}$$

C. m≠0 Solutions (Non–Zero Separation Constant Solutions)

$$r \frac{d}{dr} \left(r \frac{dR}{dr} \right) - m^2 R = 0 \qquad\qquad \frac{d^2 F}{d\phi^2} + m^2 F = 0$$

Try $R = Ar^n$ $\qquad\qquad\qquad\qquad\qquad$ $F = A_1 \sin m\phi + A_2 \cos m\phi$

$$r \frac{d}{dr} \left[n\!\!\!/ A r^n \right] - m^2 \!\!\!/ A r^n = 0$$

$$n^2 r^n - m^2 r^n = 0 \implies n = \pm m$$

$$R(r) = A_3 r^m + A_4 r^{-m}$$

$$\Phi(r, \phi) = R(r)F(\phi) = \left[A_1 \sin m\phi + A_2 \cos m\phi\right]\left[A_3 r^m + A_4 r^{-m}\right]$$
$$= A \sin m\phi r^m + B \sin m\phi r^{-m} + C \cos m\phi r^m + D \cos m\phi r^{-m}$$
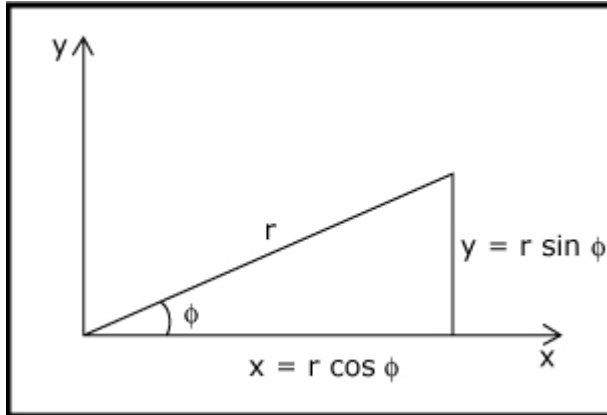
D. Selected Solutions

   m=1

$$\Phi(r, \phi) = Ar \sin \phi = Ay$$

$$\Phi(r, \phi) = Cr \cos \phi = Cx$$

$\Phi(r, \phi) = \dfrac{B \sin \phi}{r} \Rightarrow$ Line dipole oriented in y direction

$\Phi(r, \phi) = \dfrac{D \cos \phi}{r} \Rightarrow$ Line dipole oriented in x direction

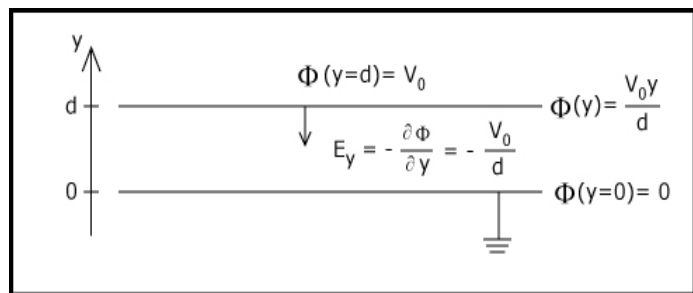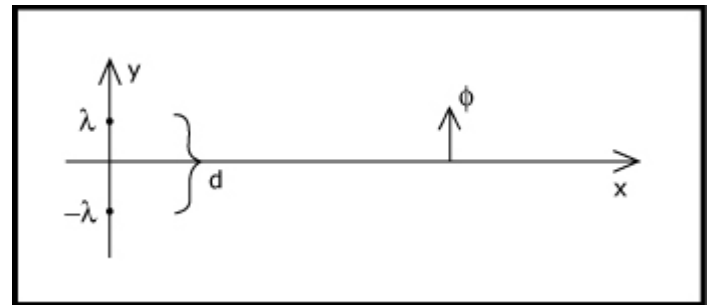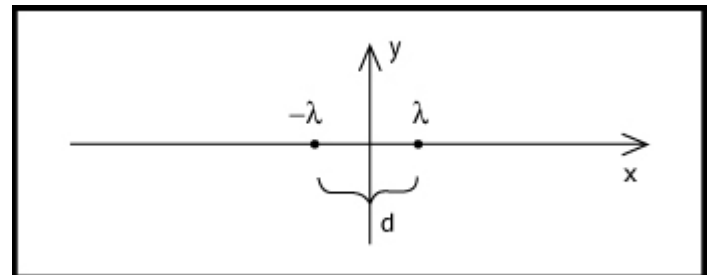<u>m=2</u>

$$\Phi = Ar^2 \sin 2\phi = 2Ar^2 \sin\phi \cos\phi = 2Axy$$
$$\text{Generally m an integer if } \Phi(\phi = 0) = \Phi(\phi = 2\pi)$$

E. Grounded Cylinder in a Uniform y Directed Electric Field
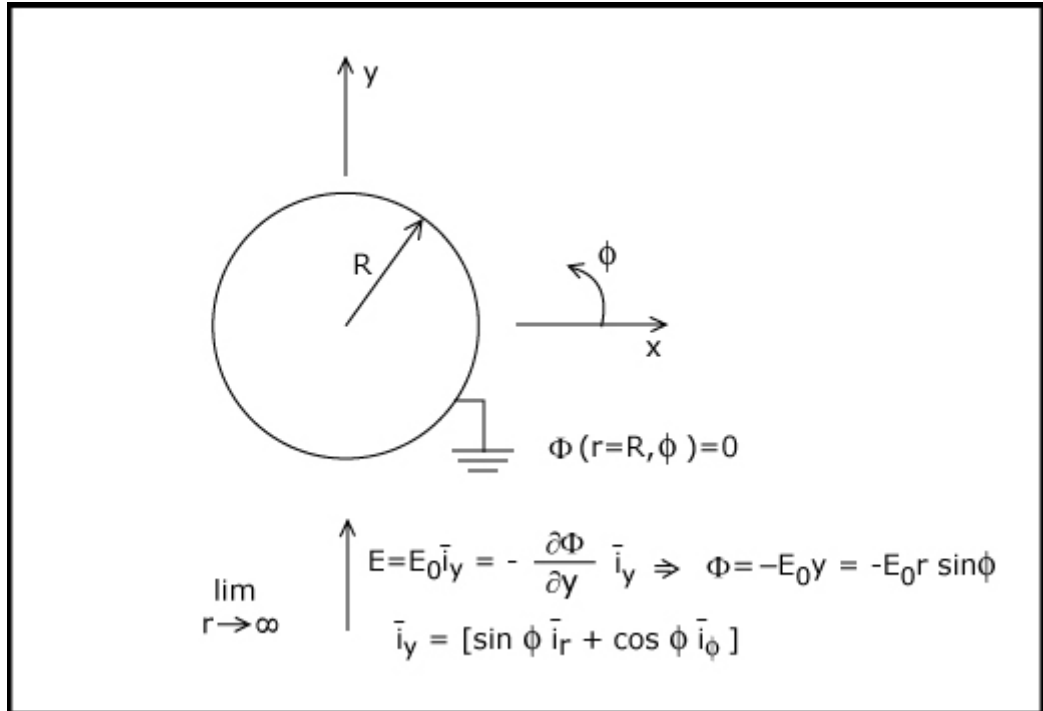
$$\Phi = \left[-E_0 r + \frac{A}{r} \right] \sin\phi \qquad r \geq R$$

$$\Phi(r = R, \phi) = 0 \implies -E_0 R + \frac{A}{R} = 0 \implies A = E_0 R^2$$

$$\Phi = -E_0 \left(r - \frac{R^2}{r} \right) \sin\phi \qquad r \geq R$$

$$\bar{E} = -\nabla\Phi = -\left[\frac{\partial\Phi}{\partial r} \bar{i}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi} \bar{i}_\phi + \cancelto{0}{\frac{\partial\Phi}{\partial z}} \bar{i}_z \right]$$

$$= E_0 \left\{ \left[1 + \frac{R^2}{r^2} \right] \sin\phi\, \bar{i}_r - \left[1 - \frac{R^2}{r^2} \right] \cos\phi\, \bar{i}_\phi \right\} \qquad r > R$$

$$\sigma_s(r = R, \phi) = \varepsilon_0 \left[E_r(r = R_+, \phi) - \cancelto{0}{E_r(r = R_-, \phi)} \right] = 2\varepsilon_0 E_0 \sin\phi$$

$$\frac{dr}{rd\phi} = \frac{E_r}{E_\phi} = -\frac{\left(1 + \frac{a^2}{r^2}\right)}{\left(1 - \frac{a^2}{r^2}\right)} \cot\phi$$

$$\Rightarrow \left(\frac{r}{a} + \frac{a}{r}\right) \sin\phi = \text{const} \text{ ———}$$

$$V = \begin{cases} 0 & r \leqslant a \\ E_0 a\left(\frac{a}{r} - \frac{r}{a}\right) \cos\phi & r \geqslant a \end{cases}$$

$$E = -\nabla V = \begin{cases} 0 & r < a \\ E_0 \left[\left(1 + \frac{a^2}{r^2}\right) \cos\phi i_r - \left(1 - \frac{a^2}{r^2}\right) \sin\phi i_\phi\right] & r > a \end{cases}$$

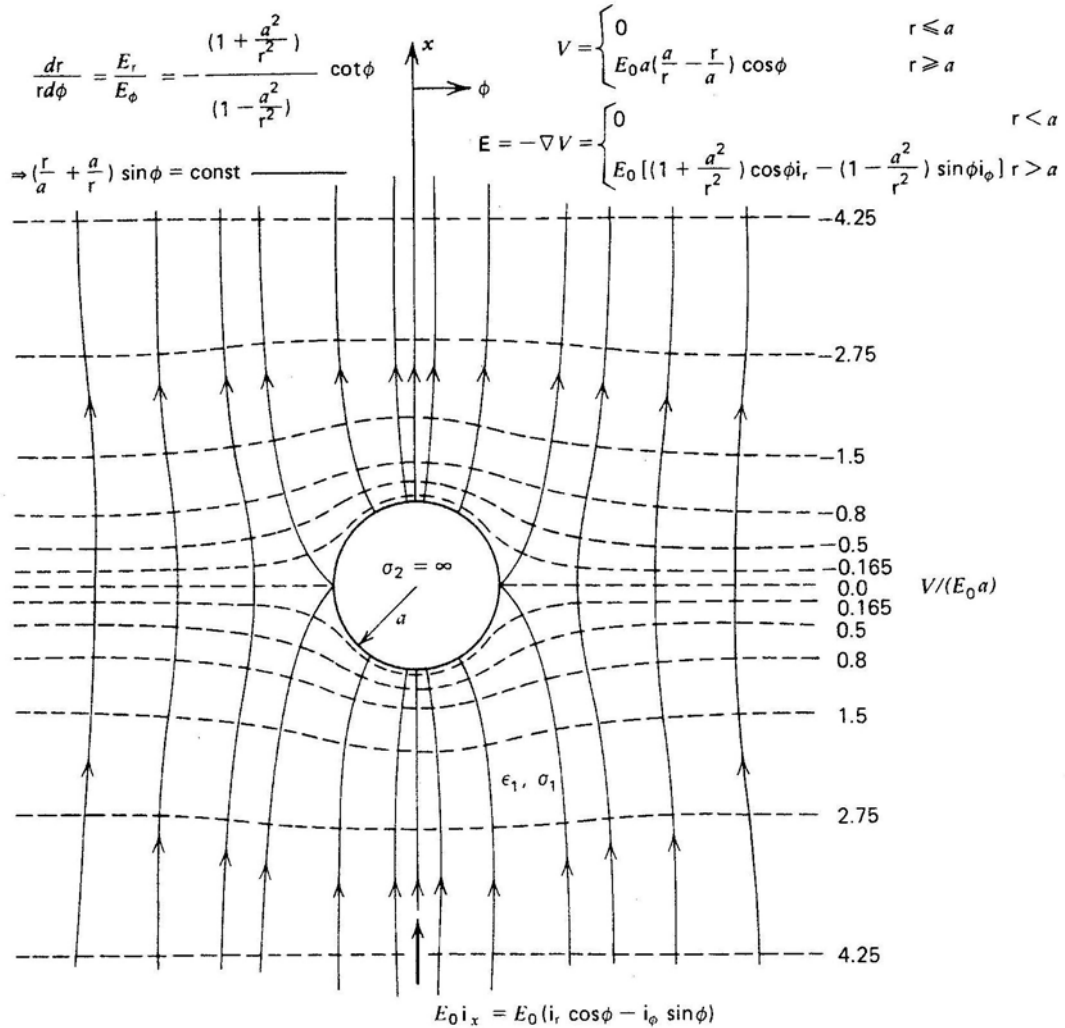$E_0 i_x = E_0 (i_r \cos\phi - i_\phi \sin\phi)$

Figure 4-8 Steady-state field and equipotential lines about a (a) perfectly conducting or (b) perfectly insulating cylinder in a uniform electric field.

Courtesy of Krieger Publishing. Used with permission.

II. Solutions to Laplace's Equation in Spherical Coordinates $\left(\frac{\partial}{\partial\phi} = 0\right)$

A. Product Solution

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Phi}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Phi}{\partial\phi^2} = 0$$

$$\Phi(r, \theta) = R(r) \Theta(\theta)$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 11
Page 5 of 8

$$\frac{\Theta\,(\theta)}{r^2}\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = 0 \qquad \left| \begin{array}{l}\text{Multiply through}\\[4pt]\text{by}\quad \dfrac{r^2}{R\,(r)\,\Theta\,(\theta)}\end{array}\right.$$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right)}_{n(n+1)} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)}_{-n(n+1)} = 0$$

$$\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right) - n\,(n+1)\;R = 0$$

Try $R = Ar^p$

$$\cancel{A}p\,(p+1)\cancel{r^p} - n\,(n+1)\cancel{A}\cancel{r^p} = 0 \;\Rightarrow\; p = n,\,-(n+1)$$

$$R\,(r) = Ar^n + Br^{-(n+1)}$$

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + n\,(n+1)\sin\theta\,\Theta = 0 \quad \text{[Legendre's Equation]}$$

In 6.641, only responsible for n=1 solution

$$\Rightarrow \Theta\,(\theta) = \cos\theta$$

$\Phi = Ar\cos\theta = Az$ is potential of uniform z directed electric field

$\Phi = \dfrac{B\cos\theta}{r^2}$ is potential of point electric dipole

n=1

$$\Phi\,(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta$$

B. Grounded Sphere in a Uniform z Directed Electric Field

$$z = r\cos\theta$$

$$\nabla z = \bar{i}_z = \nabla\,(r\cos\theta) = \bar{i}_r\,\frac{\partial}{\partial r}\,(r\cos\theta) + \bar{i}_\theta\,\frac{1}{r}\frac{\partial}{\partial\theta}\,(r\cos\theta) = \bar{i}_r\,\cos\theta - \bar{i}_\theta\,\sin\theta$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 11
Page 6 of 8

$$\bar{E} = E_0 \bar{i}_z = E_0 [\cos\theta\, \bar{i}_r - \sin\theta\, \bar{i}_\theta] = -\frac{\partial \Phi}{\partial z}\, \bar{i}_z$$

$$\lim_{r \gg R} \Phi = -E_0 z = -E_0 r \cos\theta$$
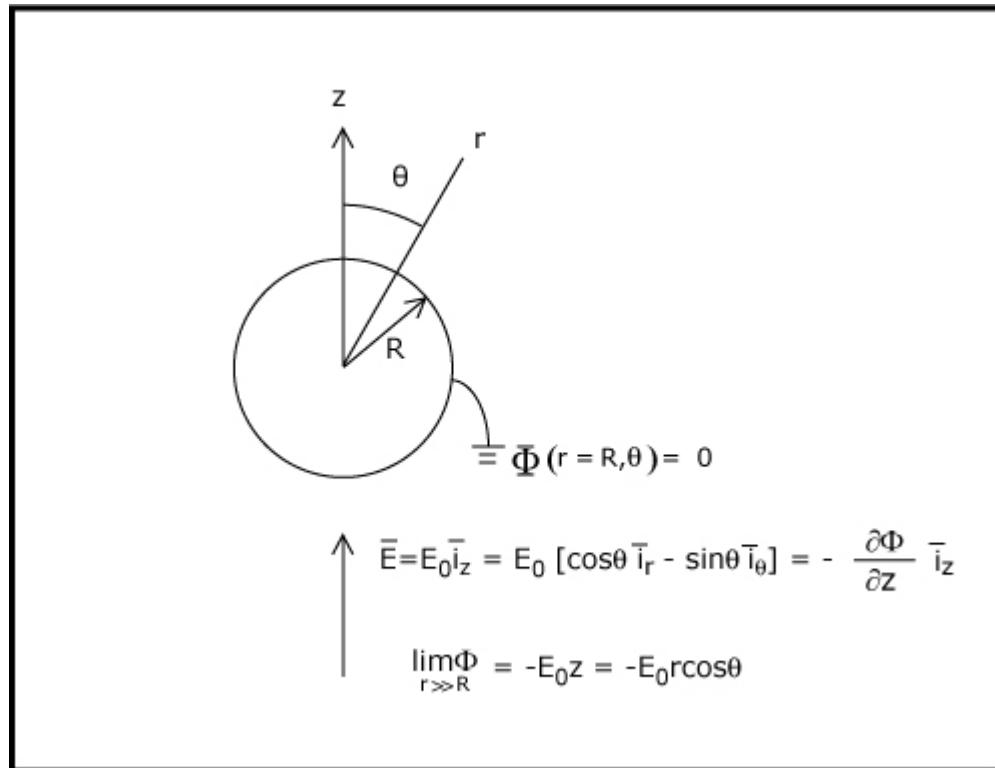
$r \geq R$

$$\Phi\,(r,\,\theta) = -E_0 r \cos\theta + \frac{A \cos\theta}{r^2}$$

$$\Phi\,(r = R,\,\theta) = 0 = \left(-E_0 R + \frac{A}{R^2}\right)\cos\theta \;\Rightarrow\; A = E_0 R^3$$

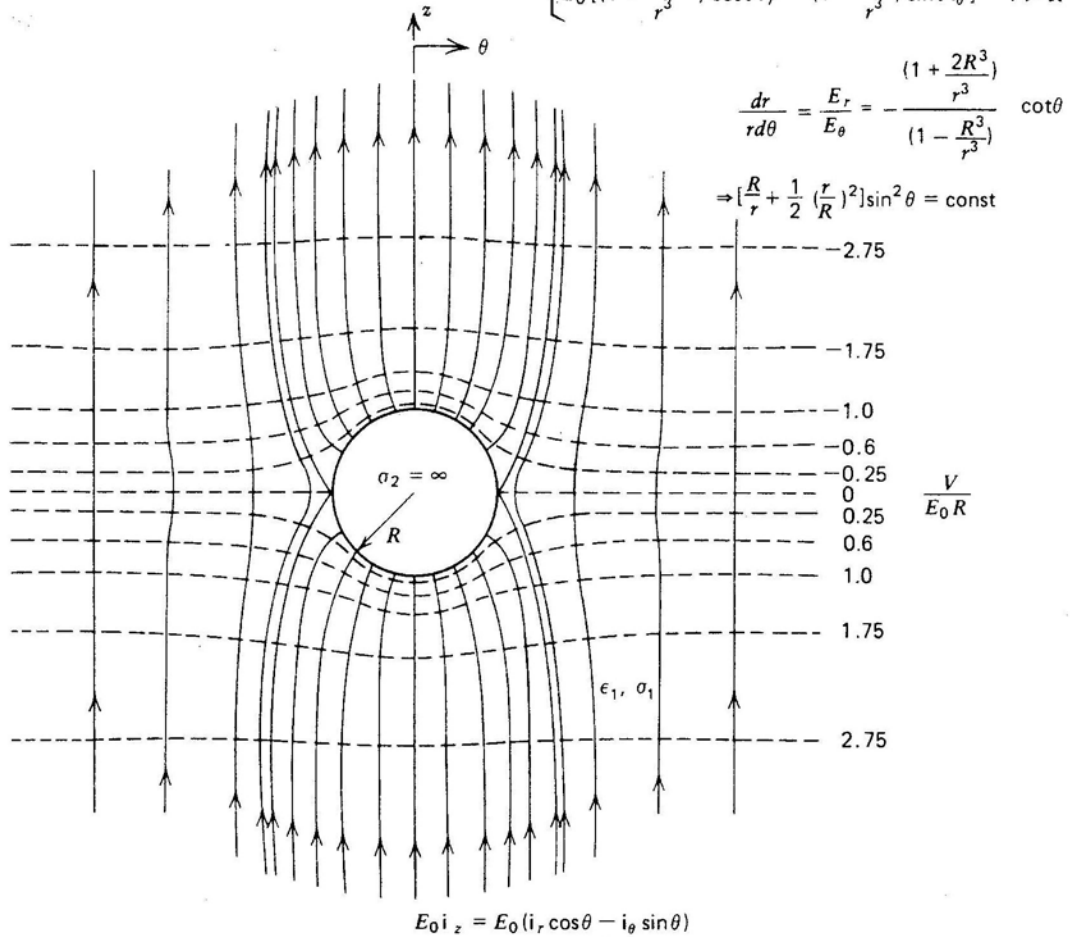$$\Phi\,(r,\,\theta) = -E_0 \left(r - \frac{R^3}{r^2}\right)\cos\theta \qquad r \geq R$$

$$\bar{E} = -\nabla\Phi = -\left[\frac{\partial \Phi}{\partial r}\,\bar{i}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\,\bar{i}_\theta\right]$$

$$= E_0 \left[\left(1 + \frac{2R^3}{r^3}\right)\cos\theta\,\bar{i}_r - \left(1 - \frac{R^3}{r^3}\right)\sin\theta\,\bar{i}_\theta\right]$$

$$\sigma_s\,(r = R,\,\theta) = \varepsilon_0 E_r\,(r = R,\,\theta) = 3\varepsilon_0 E_0 \cos\theta$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 11
Page 7 of 8

$$V = \begin{cases} 0 & r \leqslant R \\ -E_0 R \left(\dfrac{r}{R} - \dfrac{R^2}{r^2} \right) \cos \theta & r \geqslant R \end{cases}$$

$$\mathbf{E} = -\nabla V = \begin{cases} 0 & r < R \\ E_0 \left[\left(1 + \dfrac{2R^3}{r^3} \right) \cos\theta\, \mathbf{i}_r - \left(1 - \dfrac{R^3}{r^3} \right) \sin\theta\, \mathbf{i}_\theta \right] & r > R \end{cases}$$

$$\frac{dr}{r\, d\theta} = \frac{E_r}{E_\theta} = -\frac{\left(1 + \dfrac{2R^3}{r^3} \right)}{\left(1 - \dfrac{R^3}{r^3} \right)} \cot\theta$$

$$\Rightarrow \left[\frac{R}{r} + \frac{1}{2} \left(\frac{r}{R} \right)^2 \right] \sin^2\theta = \text{const}$$

$E_0 \mathbf{i}_z = E_0 (\mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta)$

Courtesy of Krieger Publishing. Used with permission.

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 11
Page 8 of 8