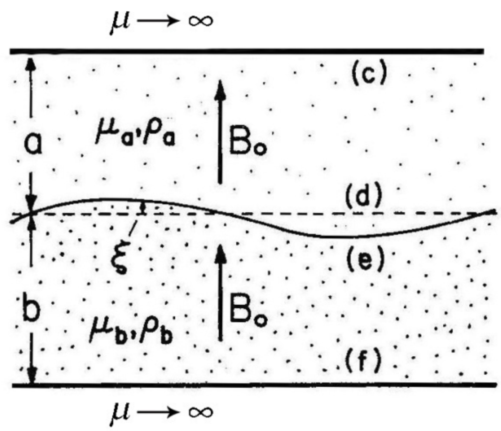
6.642 Continuum Electromechanics
Fall 2008

Lecture 8: Electrohydrodynamic and Ferrohydrodynamic instabilities

I. Magnetic Field Normal Instability



(a) Layers of magnetizable fluid are stressed by a uniform normal magnetic flux density, B_o.

A. Equilibrium $(\xi = 0)$

$$B_0 = \mu_a H_a = \mu_b H_b \qquad ; \qquad P_{oe} - P_{od} + \frac{1}{2}\left[\mu_a H_a^2 - \mu_b H_b^2\right] = 0$$

$$P_o(x) = \begin{cases} -\rho_a g x + P_{od} & x > 0 \\ -\rho_b g x + P_{oe} & x < 0 \end{cases}$$

B. Perturbations: $e^{j(\omega t - k_y y - k_z z)}$

$$\bar{H}_a = H_a \bar{i}_x + \bar{h}_a \,, \quad \bar{H}_b = H_b \bar{i}_x + \bar{h}_b$$

$$\begin{bmatrix} \hat{h}_{xc} \\ \hat{h}_{xd} \end{bmatrix} = k \begin{bmatrix} -\coth ka & \frac{1}{\sinh ka} \\ -\frac{1}{\sinh ka} & \coth ka \end{bmatrix} \begin{bmatrix} \hat{\Psi}^c \\ \hat{\Psi}^d \end{bmatrix}$$

$$\begin{bmatrix} \hat{h}_{xe} \\ \hat{h}_{xf} \end{bmatrix} = k \begin{bmatrix} -\coth kb & \frac{1}{\sinh kb} \\ -\frac{1}{\sinh kb} & \coth kb \end{bmatrix} \begin{bmatrix} \hat{\Psi}^e \\ \hat{\Psi}^f \end{bmatrix}$$

C. Boundary Conditions

$$\hat{v}_{xc} = \hat{v}_{xf} = 0$$

$$\hat{\Psi}^{c} = \hat{\Psi}^{f} = 0$$

$$v_{xd,e} = \frac{\partial \xi}{\partial t} + v_{yd,e}\frac{\partial \xi}{\partial y} + v_{zd,e}\frac{\partial \xi}{\partial z} \Rightarrow \hat{v}_{xd} = \hat{v}_{xe} = j\omega\hat{\xi}$$

$$\bar{n} = \frac{\bar{i}_x - \frac{\partial \xi}{\partial y}\bar{i}_y - \frac{\partial \xi}{\partial z}\bar{i}_z}{\left[1 + \left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2\right]} \approx \bar{i}_x - \frac{\partial \xi}{\partial y}\bar{i}_y - \frac{\partial \xi}{\partial z}\bar{i}_z$$

$$\|p\| n_i = \|T_{ij}^{e}\| n_j - \gamma \nabla \cdot \bar{n}\, n_i$$

$$\nabla \cdot \bar{n} = -\left(\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}\right)$$

$$i = x \Rightarrow n_i = n_x = 1$$

$$\|P_0 + P'\| = \|T_{xx}\| \underbrace{n_x}_{1} + \underbrace{\|T_{xy}\|}_{\text{perturbation } \mu H_x h_y} \underbrace{n_y}_{\text{perturbation}} + \|T_{xz}\| n_z + \gamma\left(\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}\right)$$

perturbation $\mu H_x h_y$, perturbation $\mu H_x h_z$: second order

Equilibrium $(\xi = 0)$

$$\|P_0\| = \|T_{xx0}\| \Rightarrow P_{od} - P_{oe} = \frac{1}{2}\left[\mu_a H_a^2 - \mu_b H_b^2\right]$$

Perturbations

$$P_d'(\xi) - P_e'(\xi) = \frac{1}{\cancel{2}}\left[\cancel{2}\mu_a H_a h_d'(\xi) - \cancel{2}\mu_b H_b h_e'(\xi)\right] - \gamma k^2 \hat{\xi}$$

$$P_d'(\xi) = P_{od}(\xi) + P_d'(0) = \left.\frac{dP_{od}}{dx}\right|_{x=0} \xi + P_d'(0) = -\rho_a g \xi + P_d'(0)$$

$$P_e'(\xi) = P_{oe}(\xi) + P_e'(0) = \left.\frac{dP_{oe}}{dx}\right|_{x=0} \xi + P_e'(0) = -\rho_b g \xi + P_e'(0)$$

$$\begin{bmatrix}\hat{p}_c\\ \hat{p}_d\end{bmatrix} = \frac{j\omega\rho_a}{k}\begin{bmatrix}-\coth ka & \dfrac{1}{\sinh ka}\\ -\dfrac{1}{\sinh ka} & \coth ka\end{bmatrix}\begin{bmatrix}\hat{v}_x^c\\ \hat{v}_x^d\end{bmatrix}$$

$$\begin{bmatrix}\hat{p}_e\\ \hat{p}_f\end{bmatrix} = \frac{j\omega\rho_b}{k}\begin{bmatrix}-\coth kb & \dfrac{1}{\sinh kb}\\ -\dfrac{1}{\sinh kb} & \coth kb\end{bmatrix}\begin{bmatrix}\hat{v}_x^e\\ \hat{v}_x^f\end{bmatrix}$$

$$\hat{p}_d = \frac{j\omega\rho_a}{k}\coth ka\hat{v}_x^d = \frac{-\omega^2\rho_a}{k}\coth ka\ \hat{\xi}$$

$$\hat{P}_e = -\frac{j\omega\rho_b}{k}\coth kb\hat{v}_x^e = \frac{\omega^2\rho_b}{k}\coth kb\ \hat{\xi}$$

$$\hat{h}_{xd} = k\coth ka\hat{\Psi}^d$$

$$\hat{h}_{xe} = -k\coth kb\hat{\Psi}^e$$

$$\bar{n}\bullet\|\bar{b}\| = 0 = \left[\bar{i}_x - \frac{\partial\xi}{\partial y}\bar{i}_y - \frac{\partial\xi}{\partial z}\bar{i}_z\right]\bullet\left[(B_a - B_b)\bar{i}_x + (\mu_a h_{xd} - \mu_b h_{xe})\bar{i}_x + (\mu_a h_{yd} - \mu_b h_{ye})\bar{i}_y\right.$$

$$\left. + (\mu_a h_{zd} - \mu_b h_{ze})\bar{i}_z\right]$$

$$B_a = B_b = B_0$$
$$\mu_a h_{xd} = \mu_b h_{xe}$$

$$\bar{n}\times\|\bar{h}\| = 0 = \left[\bar{i}_x - \frac{\partial\xi}{\partial y}\bar{i}_y - \frac{\partial\xi}{\partial z}\bar{i}_z\right]\times\left[(H_a - H_b)\bar{i}_x + (h_{xd} - h_{xe})\bar{i}_x + (h_{yd} - h_{ye})\bar{i}_y + (h_{zd} - h_{ze})\bar{i}_z\right]$$

$$= \bar{i}_z(h_{yd} - h_{ye}) - \bar{i}_y(h_{zd} - h_{ze}) + \frac{\partial\xi}{\partial y}(H_a - H_b)\bar{i}_z - \frac{\partial\xi}{\partial z}(H_a - H_b)\bar{i}_y$$

$$(h_{yd} - h_{ye}) = -\frac{\partial\xi}{\partial y}(H_a - H_b) \Rightarrow +jk_y(\hat{\Psi}_d - \hat{\Psi}_e) = jk_y(H_a - H_b)\hat{\xi}$$

$$(h_{zd} - h_{ze}) = -\frac{\partial\xi}{\partial z}(H_a - H_b) \Rightarrow +jk_z(\hat{\Psi}_d - \hat{\Psi}_e) = jk_z(H_a - H_b)\hat{\xi}$$

$$\hat{\Psi}_d - \hat{\Psi}_e = +(H_a - H_b)\hat{\xi}$$

$$\mu_a \not{k}\coth ka\hat{\Psi}_d = -\mu_b \not{k}\coth kb\hat{\Psi}_e$$

$$\hat{\Psi}_d = \frac{-\mu_b}{\mu_a}\frac{\coth kb}{\coth ka}\hat{\Psi}_e$$

$$\hat{\Psi}_e\left[-\frac{\mu_b}{\mu_a}\frac{\coth kb}{\coth ka} - 1\right] = \left(H_a - H_b\right)\hat{\xi}$$

$$\hat{\Psi}_e = -\frac{\left(H_a - H_b\right)\hat{\xi}\;\;\mu_a \coth ka}{\mu_a \coth ka + \mu_b \coth kb}$$

$$\hat{\Psi}_d = +\frac{\left(H_a - H_b\right)\hat{\xi}\;\;\mu_b \coth kb}{\mu_a \coth ka + \mu_b \coth kb}$$

D. Dispersion Equation

$$-\rho_a g\hat{\xi} - \frac{\omega^2 \rho_a}{k}\coth ka\hat{\xi} + \rho_b g\hat{\xi} - \frac{\omega^2 \rho_b}{k}\coth kb\,\hat{\xi}$$

$$= \frac{\mu_a H_a\, k \coth ka\left(H_a - H_b\right)\mu_b \coth kb\,\hat{\xi} - \mu_b H_b\, k \coth kb\left(H_a - H_b\right)\hat{\xi}\mu_a \coth ka}{\mu_a \coth ka + \mu_b \coth kb} - \gamma k^2\hat{\xi}$$

$$\frac{\omega^2}{k}\left(\rho_a \coth ka + \rho_b \coth kb\right) = \left(\rho_b - \rho_a\right)g + \gamma k^2 - \frac{B_0\left(H_a - H_b\right)\coth ka \coth kb\left(\mu_b - \mu_a\right)k}{\mu_a \coth ka + \mu_b \coth kb}$$

$$H_a = \frac{B_0}{\mu_a},\; H_b = \frac{B_0}{\mu_b} \Rightarrow H_a - H_b = B_0\left(\frac{1}{\mu_a} - \frac{1}{\mu_b}\right) = \frac{\left(\mu_b - \mu_a\right)B_0}{\mu_a \mu_b}$$

$$\frac{\omega^2}{k}\left(\rho_a \coth ka + \rho_b \coth kb\right) = g\left(\rho_b - \rho_a\right) + \gamma k^2 - \frac{B_0^2\left(\mu_b - \mu_a\right)^2\, k}{\mu_a \mu_b\left[\mu_b \tanh ka + \mu_a \tanh kb\right]}$$

E. Short Wavelength Limit $\left(ka \gg 1,\; kb \gg 1\right)$

$$\tanh ka \approx \tanh kb \approx 1$$

$$\frac{\omega^2}{k}\left(\rho_a + \rho_b\right) = g\left(\rho_b - \rho_a\right) + \gamma k^2 - \frac{B_0^2\left(\mu_b - \mu_a\right)^2 k}{\mu_a \mu_b\left(\mu_a + \mu_b\right)} = f$$

Incipience of Instability

$$f = 0$$

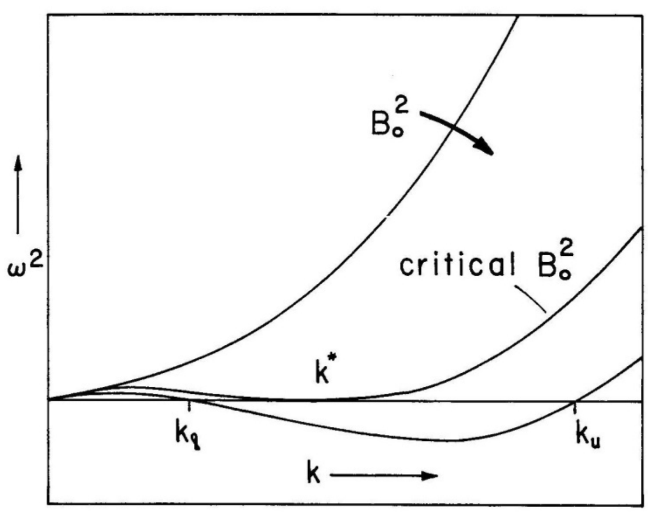
$$\frac{df}{dk} = 0 = 2\gamma k_c - \frac{B_0^2 (\mu_b - \mu_a)^2}{\mu_a \mu_b (\mu_a + \mu_b)}$$

$$k_c = \frac{B_0^2 (\mu_b - \mu_a)^2}{2\gamma \mu_a \mu_b (\mu_a + \mu_b)}$$

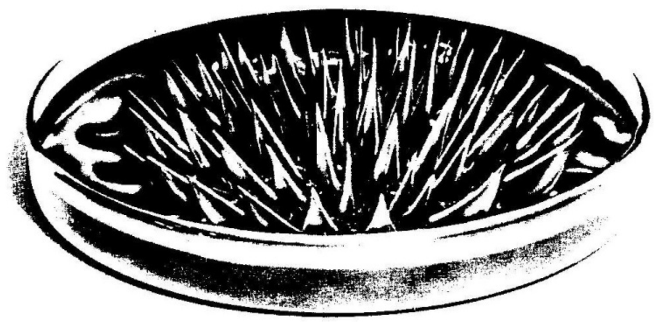
$$g(\rho_b - \rho_a) + \gamma \left[\frac{B_0^2 (\mu_b - \mu_a)^2}{2\gamma \mu_a \mu_b (\mu_a + \mu_b)} \right]^2 - \left[\frac{B_0^2 (\mu_b - \mu_a)^2}{\mu_a \mu_b (\mu_a + \mu_b)} \right]^2 \frac{1}{2\gamma} = 0$$

$$\left[\frac{B_0^2 (\mu_b - \mu_a)^2}{\mu_a \mu_b (\mu_a + \mu_b)} \right]^2 = 4g(\rho_b - \rho_a)\gamma$$

$$k_c = \frac{1}{2\gamma}\sqrt{4g(\rho_b - \rho_a)\gamma} = \sqrt{\frac{g(\rho_b - \rho_a)}{\gamma}}$$



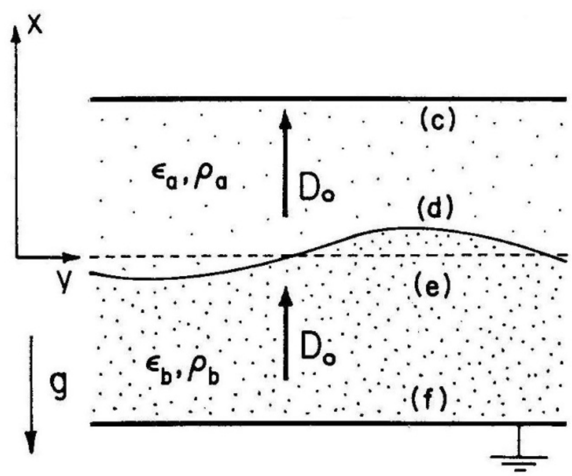
Dependence of $\omega^2(k)$ as given by Eq. 18 with B_o^2 as a parameter.



System of static fluid sprouts represents a new static equilibrium formed once planar interface in perpendicular field becomes unstable. (Courtesy of Ferrofluidics Corp., Burlington, Mass.)

II. Electric Field Normal Instability

A. Polarization Forces



Polarizable liquid layers are stressed by a normal electric displacement, D_o.

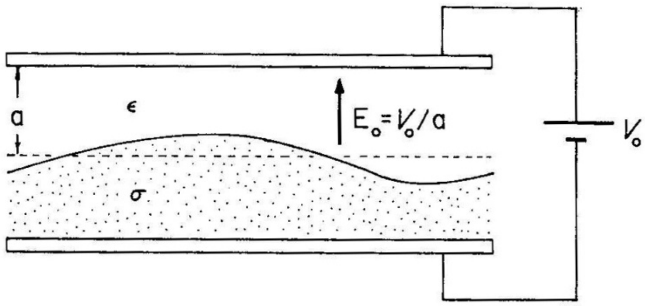
$\mu_a \rightarrow \varepsilon_a$

$\mu_b \rightarrow \varepsilon_b$

$B_0 \rightarrow D_0$

$$\frac{\omega^2}{k}\left(\rho_a \coth ka + \rho_b \coth kb\right) = g\left(\rho_b - \rho_a\right) + \gamma k^2 - \frac{D_0^2\left(\varepsilon_b - \varepsilon_a\right)^2 k}{\varepsilon_a \varepsilon_b \left[\varepsilon_b \tanh ka + \varepsilon_a \tanh kb\right]}$$

B. Perfectly conducting lower fluid $\left(\varepsilon_b \rightarrow \infty\right)$



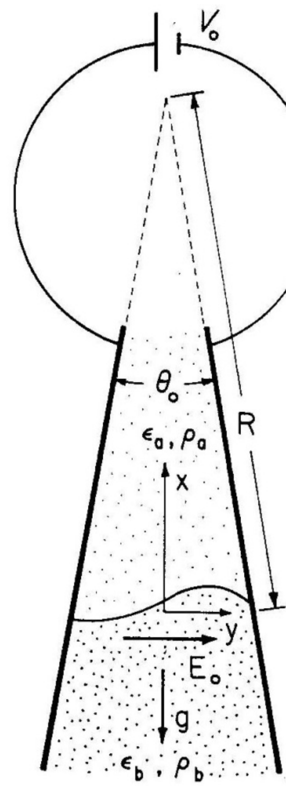
Rigid plane-parallel electrodes bound liquids having common interface. The upper liquid is insulating relative to the lower one.

Courtesy of MIT Press. Used with permission.

$$\frac{\omega^2}{k}\left(\rho_a \coth ka + \rho_b \coth kb\right) = g\left(\rho_b - \rho_a\right) + \gamma k^2 - \varepsilon_a \left(\frac{V_0}{a}\right)^2 k \coth ka$$

Nonlinear stages of surface instability caused by applying 30 kV d-c between electrode above and glycerine interface below. Insulation is mixture of air and gaseous Freon.

III. Tangential Gradient Fields



Cross section experiment with Cartesian coordinates for planar model.

A. Equilibrium

$$\bar{E} = \bar{i}_\theta \frac{V_0}{\theta_0 r} \approx \bar{i}_y \frac{V_0}{\theta_0 r}\left(1+\frac{x}{R}\right);\ r \approx R - x\ ;\ E_0 = \frac{V_0}{\theta_0 R}$$

$$\approx \bar{i}_y E_0 \left(1+\frac{x}{R}\right)$$

$$\lim\ \Delta \to \infty \qquad P_{oa}(x) = -\rho_a g x + P_{oa}$$

$$P_{ob}(x) = -\rho_b g x + P_{ob}$$

$$P_{ob} - P_{oa} + \|T_{xx}\| = 0$$

$$T_{xx} = -\frac{1}{2}\varepsilon E_y^2 = -\frac{1}{2}\varepsilon E_0^2$$

$$\|T_{xx}\| = -\frac{1}{2}(\varepsilon_a - \varepsilon_b)E_0^2 = P_{oa} - P_{ob}$$

B. Perturbations

$$\begin{bmatrix} \hat{e}_x^\alpha \\ \hat{e}_x^\beta \end{bmatrix} = k \begin{bmatrix} -\coth k\Delta & \dfrac{1}{\sinh k\Delta} \\ -\dfrac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{\Phi}^\alpha \\ \hat{\Phi}^\beta \end{bmatrix}$$

$$\lim\ \Delta \to \infty$$

$$\hat{e}_{xa} = k\,\hat{\Phi}_a$$

$$\hat{e}_{xb} = -k\,\hat{\Phi}_b$$

$$\begin{bmatrix} \hat{p}^\alpha \\ \hat{p}^\beta \end{bmatrix} = \frac{j\omega\rho}{k} \begin{bmatrix} -\coth k\Delta & \dfrac{1}{\sinh k\Delta} \\ -\dfrac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \hat{v}_x^\alpha \\ \hat{v}_x^\beta \end{bmatrix}$$

$$\hat{v}_x^\alpha = \hat{v}_x^\beta = j\omega\hat{\xi}$$

$$\hat{P}_a = \frac{j\omega\rho_a}{k}\,\hat{v}_{xa} = -\frac{\omega^2\rho_a}{k}\hat{\xi}$$

$$\hat{P}_b = -\frac{j\omega\rho_b}{k}\,\hat{v}_{xb} = \frac{\omega^2\rho_b}{k}\hat{\xi}$$

C. Boundary Conditions

$$\bar{n} = \bar{i}_x - \frac{\partial\xi}{\partial y}\,\bar{i}_y - \frac{\partial\xi}{\partial z}\,\bar{i}_z$$

$$\bar{n}\times\left\|\bar{E}\right\| = 0 \Rightarrow \left[\bar{i}_x - \frac{\partial\xi}{\partial y}\,\bar{i}_y - \frac{\partial\xi}{\partial z}\,\bar{i}_z\right]\times\left[\left(e_{xa}-e_{xb}\right)\bar{i}_x + \left(e_{ya}-e_{yb}\right)\bar{i}_y + \left(e_{za}-e_{zb}\right)\bar{i}_z\right] = 0$$

$$\bar{i}_z\left(e_{ya}-e_{yb}\right) - \bar{i}_y\left(e_{za}-e_{zb}\right) = 0$$

$$\left.\begin{array}{l} e_{ya} = e_{yb} \Rightarrow jk_y\,\hat{\Phi}_a = jk_y\,\hat{\Phi}_b \\ e_{za} = e_{zb} \Rightarrow jk_z\,\hat{\Phi}_a = jk_z\,\hat{\Phi}_b \end{array}\right\} \Rightarrow \hat{\Phi}_a = \hat{\Phi}_b \equiv \hat{\Phi}$$

$$\bar{n}\cdot\left\|\varepsilon\bar{E}\right\| = 0 \Rightarrow \left[\bar{i}_x - \frac{\partial\xi}{\partial y}\,\bar{i}_y - \frac{\partial\xi}{\partial z}\,\bar{i}_z\right]\cdot\Big[E_0\left(\varepsilon_a-\varepsilon_b\right)\bar{i}_y + \left(\varepsilon_a e_{xa}-\varepsilon_b e_{xb}\right)\bar{i}_x + \left(\varepsilon_a e_{ya}-\varepsilon_b e_{yb}\right)\bar{i}_y$$
$$+\left(\varepsilon_a e_{za}-\varepsilon_b e_{zb}\right)\bar{i}_z\Big] = 0$$

$$\varepsilon_a\hat{e}_{xa} - \varepsilon_b\hat{e}_{xb} + jk_y\hat{\xi}E_0\left(\varepsilon_a-\varepsilon_b\right) = 0$$

$$k\,\hat{\Phi}\left(\varepsilon_a+\varepsilon_b\right) + jk_y\,E_0\left(\varepsilon_a-\varepsilon_b\right)\hat{\xi} = 0$$

$$\hat{\Phi} = \frac{-jk_y\,E_0\left(\varepsilon_a-\varepsilon_b\right)\hat{\xi}}{k\left(\varepsilon_a+\varepsilon_b\right)}$$

$$\left\|p\right\|n_i = \left\|T_{ij}\right\|n_j + \gamma\left[\frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\right]n_i$$

$$i = x\,,\; n_x = 1$$

$$\left\|\hat{p}\right\| = \left\|\hat{T}_{xx}\right\| + \left\|T_{xy}\right\|\hat{n}_y + \left\|T_{xz}\right\|\hat{n}_z - \gamma k^2\hat{\xi}$$

$$T_{xx} = \frac{1}{2}\varepsilon\left(E_x^2 - E_y^2 - E_z^2\right) = -\frac{1}{2}\varepsilon\left[E_{y0}+e_y\right]^2 = -\frac{1}{2}\varepsilon\left[E_{y0}^2 + 2E_0 e_y\right]$$

$$T_{xy} = \varepsilon E_{y0}\,e_x$$

$$T_{xz} = \varepsilon\,e_x\,e_z$$

$\left\| T_{xy} \right\| n_y$ second order

$\left\| T_{xz} \right\| n_z$ third order

$$T_{xx}(\xi) = -\frac{1}{2}\varepsilon \frac{d}{dx}\left(E_{y0}^2\right)\Big|_{x=0} \xi - \varepsilon E_0 e_y$$

$$= -\varepsilon E_0 \frac{dE_0}{dx} \xi - \varepsilon E_0 e_y$$

$$\hat{T}_{xx} = -\varepsilon E_0 \frac{dE_0}{dx} \hat{\xi} - \varepsilon E_0 \hat{e}_y$$

$$= -\varepsilon E_0 \frac{dE_0}{dx} \hat{\xi} - jk_y \hat{\Phi} \varepsilon E_0$$

$$\left\| \hat{T}_{xx} \right\| = -(\varepsilon_a - \varepsilon_b) E_0 \frac{dE_0}{dx} \hat{\xi} - jk_y E_0 \hat{\Phi} (\varepsilon_a - \varepsilon_b)$$

$$\left\| \hat{p}_T \right\| = \left\| \frac{dp_0}{dx} \hat{\xi} + \hat{p} \right\| = \hat{P}_a - \hat{P}_b - g(\rho_a - \rho_b)\hat{\xi}$$

$$\hat{P}_a - \hat{P}_b - g(\rho_a - \rho_b)\hat{\xi} = -(\varepsilon_a - \varepsilon_b) E_0 \frac{dE_0}{dx} \hat{\xi} - jk_y \hat{\Phi} E_0 (\varepsilon_a - \varepsilon_b) - \gamma k^2 \hat{\xi}$$

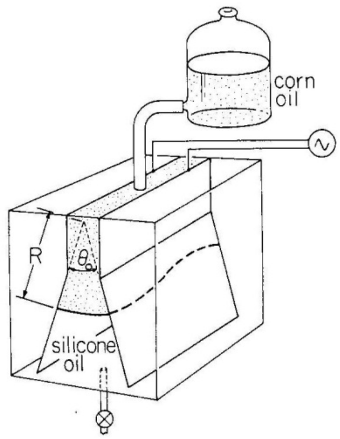
$$-\frac{\omega^2}{k}(\rho_a + \rho_b)\hat{\xi} - g(\rho_a - \rho_b)\hat{\xi} =$$

$$-(\varepsilon_a - \varepsilon_b) E_0 \frac{dE_0}{dx} \hat{\xi} - \gamma k^2 \hat{\xi} - \frac{jk_y E_0 (\varepsilon_a - \varepsilon_b)(-jk_y E_0)(\varepsilon_a - \varepsilon_b)\hat{\xi}}{k(\varepsilon_a + \varepsilon_b)}$$

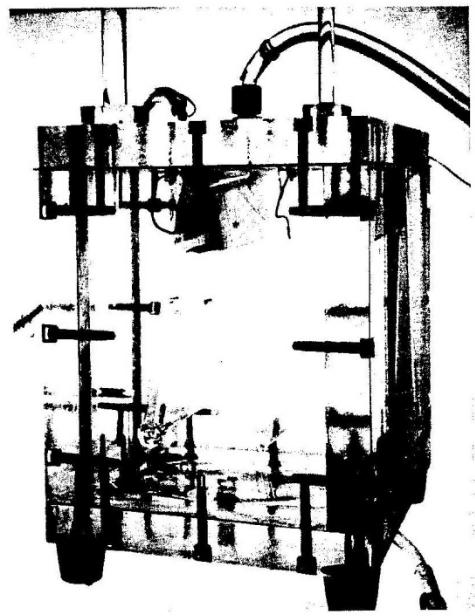
$$\frac{\omega^2}{k}(\rho_a + \rho_b) = g(\rho_b - \rho_a) + \gamma k^2 + (\varepsilon_a - \varepsilon_b) E_0 \underbrace{\frac{dE_0}{dx}}_{\frac{E_0}{R}} + \frac{k_y^2 E_0^2 (\varepsilon_a - \varepsilon_b)^2}{k(\varepsilon_a + \varepsilon_b)}$$

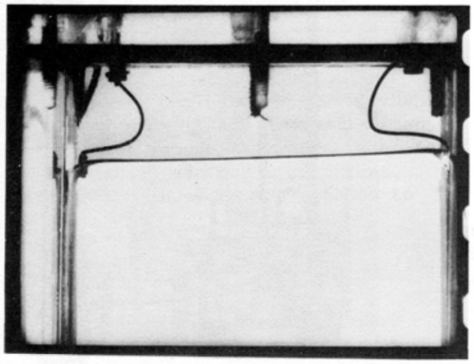
Uniform tangential field always stabilizes.
Gradient field stabilizes, if higher permittivity fluid is in stronger electric field. System stable even if heavier fluid above if

$$(\varepsilon_a - \varepsilon_b) E_0 \frac{dE_0}{dx} > g(\rho_a - \rho_b)$$



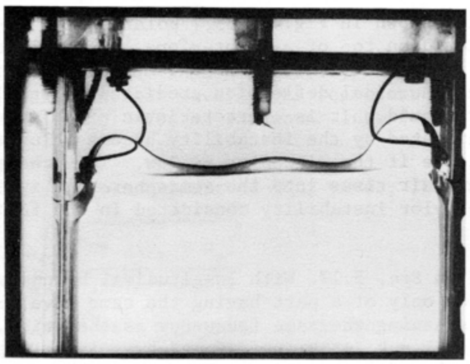
Heavy liquid is stabilized on top of lighter fluid by means of polarization forces induced by applying potential difference to the diverging glass plates. These plates have a thin transparent coating that renders them conducting.



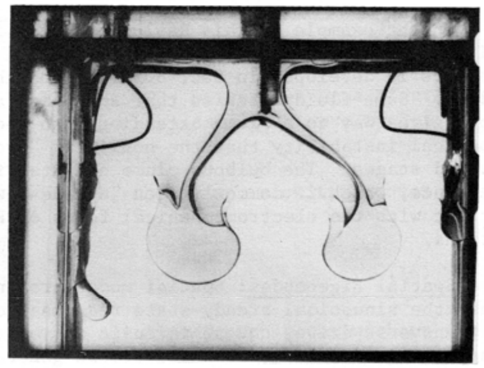


(a)

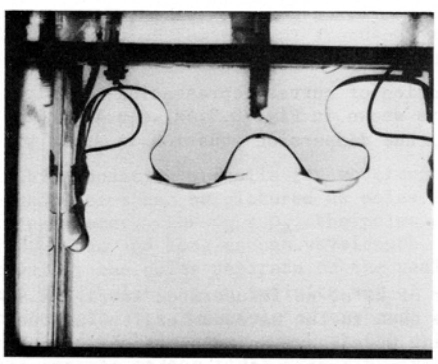
Side view of apparatus.
(a) Equilibrium with field on.
(b)-(e) Sequential view of developing instability.
(From Complex Waves II, Reference 11, Appendix C.)



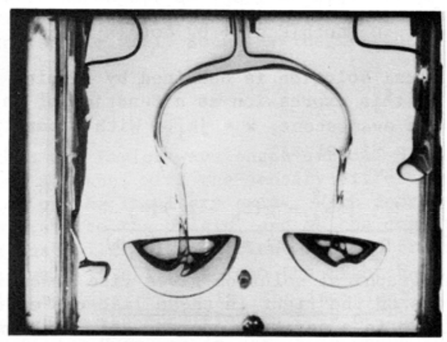
(b)



(d)



(c)



(e)