Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.685 Electric Machines

Problem Set 5 Solutions October 14, 2013

Problem 1: The six slot, two pole winding pattern is shown in cartoon form in Figure 1. If a given current in Phase A produces a peak flux density of one unit (whatever that unit might be), that implies a 'jump' of flux density of two of that unit across the location of Phase A (angle of zero). If the winding is driven by a balanced current set, at time $t = 0$, $I_a = I$, $I_b = I_c = -I/2$. Thus we expect to see a flux density distribution with 'jumps' of +2, +1, -1, -2, -1, +1, +2,... at angles of 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, $5\pi/3$, 2π, ... This is shown in Figure 2.

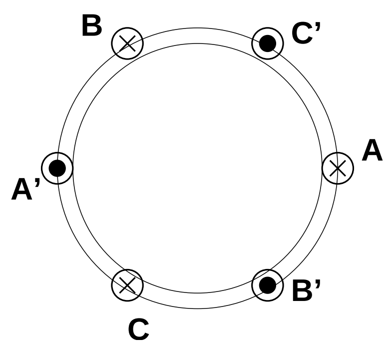

Figure 1: Cartoon of One Slot/Pole/Phase Winding

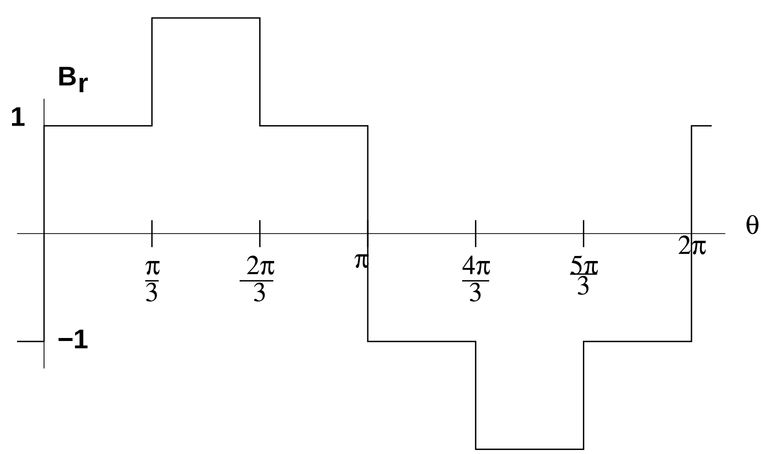

Figure 2: Expected Flux Density Pattern

To reconstruct this pattern as a Fourier Series, see that we can write the patterns for each of the three phases as:

$$B_a = \sum_n \frac{4}{n\pi} \sin(n\theta)\, I_a$$

$$B_b = \sum_n \frac{4}{n\pi} \sin\left(n(theta - \frac{2\pi}{3})\right) I_b$$
$$B_c = \sum_n \frac{4}{n\pi} \sin\left(n(theta + \frac{2\pi}{3})\right) I_c$$

where $n = [1571113...]$ This has been programmed into a Matlab script which is attached, and it produces the picture of Figure 3.

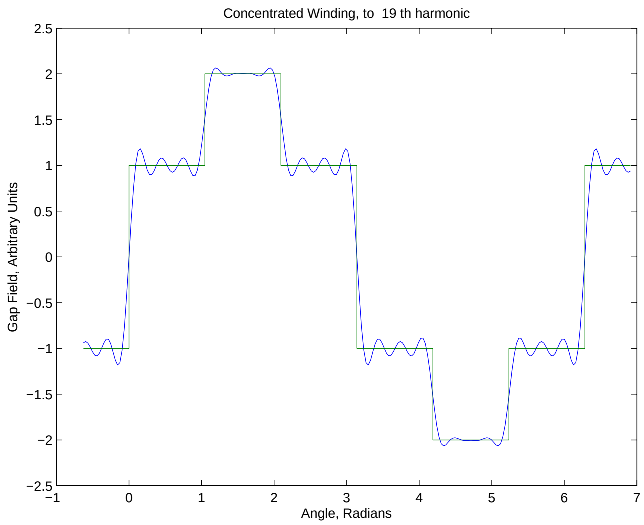

Figure 3: Reconstructed and Expected Waveform

Problem 2: The four variations on winding patterns are shown in Figure 4

For a 24 slot, two pole (p=1) machine, slot angle is $\gamma = \frac{\pi}{12} = 15°$. All variants of the 'ordinary' winding will have the same breadth factor:

$$k_{bn} = \frac{\sin(nm\frac{\gamma}{2})}{m\sin(n\frac{\gamma}{2})}$$

For full-pitch, of course, the pitch factor is always one. For the fractional pitch windings, $\alpha = \frac{5\pi}{6}$ or $\alpha = \frac{2\pi}{3}$ and, of course:

$$k_{pn} = \sin\left(n\frac{\alpha}{2}\right)$$

and $k_{wn} = k_{pn}k_{bn}$.

For the concentric winding, with coils of turns ratio 1, 2 and 3, the coil pitches are: 7, 9 and 11 slots, respectively. The winding factor can be described as:

$$k_{wn} = \frac{\sum_{k=1}^{N_{\text{coils}}} N_c \sin(n\frac{\alpha_n}{2})}{\sum_{k=1}^{N_{\text{coils}}} N_c}$$

Slot #	Fill Pitch Top	Fill Pitch Bot	5/6 Pitch Top	5/6 Pitch Bot	2/3 Pitch Top	2/3 Pitch Bot	Concentric Winding A	A'	B	B'	C	C'
1	A	A	A	B'	A	B'	1			2		
2	A	A	A	B'	A	B'	2			1		
3	A	A	A	A	A	B'	3					
4	A	A	A	A	A	B'	3					
5	C'	C'	C'	A	C'	A	2					1
6	C'	C'	C'	A	C'	A	1					2
7	C'	C'	C'	C'	C'	A						3
8	C'	C'	C'	C'	C'	A						3
9	B	B	B	C'	B	C'			1			2
10	B	B	B	C'	B	C'			2			1
11	B	B	B	B	B	C'			3			
12	B	B	B	B	B	C'			3			
13	A'	A'	A'	B	A'	B		1	2			
14	A'	A'	A'	B	A'	B		2	1			
15	A'	A'	A'	A'	A'	B		3				
16	A'	A'	A'	A'	A'	B		3				
17	C	C	C	A'	C	A'		2			1	
18	C	C	C	A'	C	A'		1			2	
19	C	C	C	C	C	A'					3	
20	C	C	C	C	C	A'					3	
21	B'	B'	B'	C	B'	C				1	2	
22	B'	B'	B'	C	B'	C				2	1	
23	B'	B'	B'	B'	B'	C				3		
24	B'	B'	B'	B'	B'	C				3		

Figure 4: Winding Patterns

Here, this becomes:

$$k_{wn} = \frac{1}{6}\left(\sin(n\frac{7\pi}{12}) + 2\sin(n\frac{9\pi}{12}) + 3\sin(n\frac{11\pi}{12})\right)$$

The attached Matlab script runs out the numbers, and here they are, reformatted:

```
Harmonic Orders
n =  1     5     7    23    25
m=4 Breadth Factors
kb = 0.9577    0.2053   -0.1576   -0.9577   -0.9577
5/6 pitch winding: pitch factor
kpa = 0.9659    0.2588    0.2588   -0.9659    0.9659
Resulting Winding Factor
kwa = 0.9250    0.0531   -0.0408    0.9250   -0.9250
2/3 pitch winding: pitch factor
kpb = 0.8660   -0.8660    0.8660   -0.8660    0.8660
Resulting Winding Factor
kwb = 0.8294   -0.1778   -0.1365    0.8294   -0.8294
Winding Factor for Concentrated Winding
kwc = 0.9359    0.1039    0.0253    0.9359   -0.9359
```

Problem 3: 1. The number of 'slots per pole' is $40/4 = 10$, so if the winding is 'short pitched' by one, the coil throw is 9. The electrical slot angle is $\gamma = p\frac{2\pi}{N_s} = \frac{4\pi}{40} \approx .314$ radians or $18°$. The coil throw angle is $\alpha = 180° - 18° = 162°$.

2. The pitch factor is:
$$k_p = \sin \frac{162^\circ}{2} \approx .9877$$
Since the number of slots per pole per phase is $m = 2$, the breadth factor is:
$$k_b = \frac{\sin 2 \times \frac{18^\circ}{2}}{2 \times \sin \frac{18^\circ}{2}} \approx .9877$$
Then the space fundamental flux density is:
$$B_r = \mu_0 \frac{5}{2} \frac{4}{\pi} \frac{NI}{2pg} = \mu_0 \times \frac{5}{2} \times \frac{4}{\pi} \times \frac{150 \times 50}{2 \times 2 \times .005} \times .9877 \times .9877 \approx 1.482T$$

---

3. To Find the amplitudes and direction of rotation of each of the space harmonics through order 41, We will let MATLAB do the heavy lifting. The magnetic field intensity in the air-gap can be written in compact form as:
$$H_r = \sum_{k=0}^{4} \sum_{n(odd)} \frac{4}{n\pi} \frac{NI}{2pg} k_{pn} k_{bn} \sin n \left(p\phi - k\frac{2\pi}{5} \right) \cos \left(\omega t - k\frac{2\pi}{5} \right)$$
Note here we are summing all five phases using the index k. Working through this we find the important sum has many elements that cancel, leaving:
$$\begin{aligned} & \sum_{k=0}^{4} \sin n \left(p\phi - k\frac{2\pi}{5} \right) \cos \left(\omega t - k\frac{2\pi}{5} \right) \\ = & \sum_{k=0}^{4} \frac{1}{2} \sin \left(np\phi + \omega t - (n+1)\frac{2\pi}{5} \right) + \frac{1}{2} \sin \left(np\phi - \omega t - (n-1)\frac{2\pi}{5} \right) \\ = & \frac{5}{2} \sin \left(np\phi \mp \omega t \right) \end{aligned}$$
where the upper sign holds for $n = 10 \times \text{integer} + 1$ and the lower sign holds for $n = 10 \times \text{integer} - 1$. All of the harmonics for which n does not meet this condition add up to zero.
A script that carries out these calculations and that does the plot (next part) is appended. Here are some of the answers provided by that script:
```
Harmonic, pitch, breadth, amplitude
   1  0.9877  0.9877  1.463
  11  0.1564  -0.1564  0.003
  21  -0.9877  -0.9877  0.070
  31  -0.1564  0.1564  0.001
  41  0.9877  0.9877  0.036
   9  0.1564  0.1564  0.004
  19  0.9877  -0.9877  0.077
  29  -0.1564  -0.1564  0.001
  39  -0.9877  0.9877  0.038
```

4. Adding up the field harmonics is just a little bit more than simply adding up a bunch of terms as indicated above, and the MATLAB script appended does this, but it is important to remember that, once we have included provision for the pitch factor, that the Fourier Series includes an additional sign factor $\sin(n-1)\frac{\pi}{2}$. If you forgot this factor, your plot will look wrong. A plot for this is shown in Figure 5.

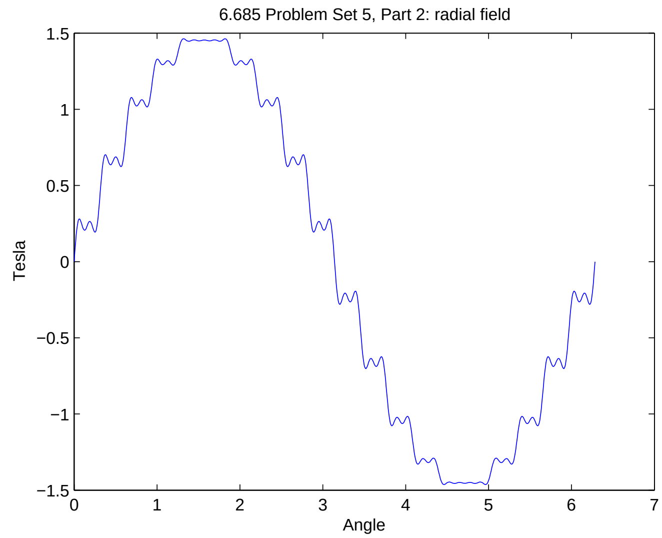

Figure 5: Radial Flux Density from Armature at $t = 0$

5. Assuming the amplitude of the space fundamental flux wave (provided by the field winding) is 1.25 T, and that this is a 60 Hz machine, flux linked by the winding is, per turn:

$$\Phi = \frac{2R\ell B_1}{p} k_p k_b \approx 0.1219 Wb$$

Then voltage is $V = \omega N \Phi = 377 \times 150 \times 0.1219 \approx 6725V$ peak. This is about 4755 Volts, RMS.

To get phase to phase voltage, note that for a five phase machine it is not just the square root of two times line-neutral. In fact, there are two different line-line voltages Since the phase spacing in a five phase system is $\xi\frac{360}{5} = 72°$, we can calculate the adjacent phase voltage difference:

$$V_{ll(\text{adjacent})} = V_{ln} 2 \sin\frac{\xi}{2} \approx 1.176 V_{ln}$$

The second line-line voltage is for phases that are not adjacent, and it is:

$$V_{ll(\text{non-adjacent})} = V_{ln} 2 \sin\frac{2\xi}{2} \approx 1.902 V_{ln}$$

These two line-line voltages are about 7,908 and 12,791 volts, peak, or 5,592 and 9,045 volts, RMS, respectively.

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6. If the rated current of this machine is 100 A (peak), what is the *per-unit* synchronous reactance of this machine?

With a peak value of 100 A, the stator would produce a (peak) flux density of $B_a = 2 \times 1.463 \approx 2.926T$, indicating a per-unit reactance of

$$x_d = \frac{2.926}{1.25} \approx 2.34\text{per-unit}$$

1 Appendix: Code

```
% 6.685 2013 Problem Set 5, Problem 1
% this is magnetic field of a concentrated stator

th = -.2*pi:pi/100:2.2*pi;        % range of angle to be used
n = [1];                          % harmonics
for k = 1:3;
    n = [n 6*k-1 6*k+1];
end

C1 = (4/pi)./ n;                  % phase A coefficients
C2 = (-2/pi) ./ n;                % phases B and C coefficients

B = zeros(size(th));              % space for the answer
B1 = zeros(size(th));

for k = 1:length(n)
    B = B + C1(k) .* sin(n(k) .* th) + C2(k) .* sin(n(k) .* (th - 2*pi/3))...
        + C2(k) .* sin(n(k) .* (th + 2*pi/3));
end

% can we construct the actual?
X = [min(th) 0 0 pi/3 pi/3 2*pi/3 2*pi/3 pi pi 4*pi/3 4*pi/3 5*pi/3 5*pi/3 2*pi 2*pi max(th)];
Y = [-1 -1 1 1 2 2 1 1 -1 -1 -2 -2 -1 -1 1 1];
titext = sprintf('Concentrated Winding, to %3.0f th harmonic',max(n));
figure(1)
plot(th, B, X, Y)
title(titext)
ylabel('Gap Field, Arbitrary Units')
xlabel('Angle, Radians')
```

```
% 6.685 Fall 2013 Problem Set 5, Problem 2

%24 slot, p=1 winding
n = [1 5 7 23 25];            % consider these orders
gama = pi/12;                 % slot pitch
m = 4;                        % slots/pole/phase

fprintf('Harmonic Orders\n')
n
fprintf('m=4 Breadth Factors\n')
kb = sin(n .* m*gama/2) ./ (m .* sin(n .* gama/2))

% now we must consider different windings
% full pitch has pitch factor of one
% 5/6 pitch: short two slots
alf = pi-2*gama;
fprintf('5/6 pitch winding: pitch factor\n')
kpa = sin(n .* alf/2)
fprintf('Resulting Winding Factor\n')
kwa = kb .* kpa
% now two thirds: short four slots
alf = pi-4*gama;
fprintf('2/3 pitch winding: pitch factor\n')
kpb = sin(n .* alf/2)
fprintf('Resulting Winding Factor\n')
kwb = kb .* kpb
% now the concentrated winding:
alf1 = 7*pi/12;
alf2 = 9*pi/12;
alf3 = 11*pi/12;
fprintf('Winding Factor for Concentrated Winding\n')
kwc = (sin(n .* alf1/2) + 2 .* sin(n .* alf2/2) + 3 .* sin(n .* alf3/2))./ 6
```

```
% 6.685 Problem Set 5, Problem 2, 2011

% This is a five phase machine

muzero = pi*4e-7;
ns = 40;         % number of stator slots
p = 2;           % number of pole pairs
N_a = 150;       % number of stator turns
R = .2;          % rotor radius
l = .5;          % active length
nsp = 1;         % short pitched by this amount
m = 2;           % did this by hand: slots/pole/phase
g = .005;        % magnetic gap
I_p = 50;       % peak amplitude of stator current

% First, find the useful harmonic orders

npos = 1:10:41;  % these are the positive going harmonic orders
nneg = 9:10:39;  % and these are the negative going harmonic orders

gama = 2*pi*p/ns;  % electrical angle between slots
alfa = pi - nsp*gama;  % electrical coil throw

kpp = sin(npos .* alfa/2);  % pitch factors for positive going harmonics
kpn = sin(nneg .* alfa/2);  % same for reverse going harmonics

kbp = sin(npos .* m*gama/2) ./ (m .* sin(npos .* gama/2));
kbn = sin(nneg .* m*gama/2) ./ (m .* sin(nneg .* gama/2));

kwp = kpp .* kbp;    % these are the winding factors
kwn = kpn .* kbn;

% now to find the peak amplitudes of the space harmonic fields

Brp = muzero*(4/pi)*(5/2) * (N_a * I_p / (2*p*g)) .* sin(npos .* pi/2) .* kwp ./ npos;
Brn = muzero*(4/pi)*(5/2) * (N_a * I_p / (2*p*g)) .* sin(nneg .* pi/2) .* kwn ./ nneg;

% now to generate the fields at t=0:

th = 0:pi/500:2*pi;
Br = zeros(size(th));
    fprintf('Harmonic, pitch, breadth, amplitude\n');

for k = 1:length(npos)
    Br = Br + Brp(k) .* sin(npos(k) .* th);
```

```
    fprintf('%4.0f  %5.4f  %5.4f  %5.3f\n', npos(k), kpp(k), kbp(k), Brp(k));
end

for k = 1:length(nneg)
    Br = Br + Brn(k) .* sin(nneg(k) .* th);
        fprintf('%4.0f  %5.4f  %5.4f  %5.3f\n', nneg(k), kpn(k), kbn(k), Brn(k));
end

% now we are all done: just need to plot


figure(1)
plot(th, Br)
title('6.685 Problem Set 5, Part 2: radial field')
ylabel('Tesla')
xlabel('Angle')
```

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