Lecture 16: Type II Superconductors

Outline

1. **A Superconducting Vortex**
2. **Vortex Fields and Currents**
3. **General Thermodynamic Concepts**
   - **First and Second Law**
   - **Entropy**
   - **Gibbs Free Energy (and co-energy)**
4. **Equilibrium Phase diagrams**
5. **Critical Fields**

November 3, 2005

Fluxoid Quantization and Type II Superconductors

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Please see: Figure 6.1, page 259, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

The Vortex State

$$\langle B \rangle = n_V \Phi_V$$

n_V is the areal density of vortices, the number per unit area.

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Please see: "A current-carrying type II superconductor in the mixed state" from http://phys.kent.edu/pages/cep.htm

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Please see: Figure 6.2a, page 262, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Quantized Vortices

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Please see: Figure 6.2b, page 262, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Fluxoid Quantization along C_1

$$\mathrm{n}\Phi_o = \oint_{C_1} \mu_o \lambda^2 \mathbf{J}_\mathrm{S} \cdot d\mathbf{l} + \int_{S_1} \mathbf{B} \cdot d\mathbf{s}$$

But along the hexagonal path C_1 **B** is a mininum, so that **J** vanishes along this path.

Therefore, $\mathrm{n}\Phi_o = \int_{S_1} \mathbf{B} \cdot d\mathbf{s}$

And experiments give n = 1, so each vortex has one flux quantum associated with it.

Along path C_2, $\Phi_o = \oint_{C_2} \mu_o \lambda^2 \mathbf{J}_\mathrm{S} \cdot d\mathbf{l} + \int_{S_2} \mathbf{B} \cdot d\mathbf{s}$

For small C_2, $\Phi_o = \lim_{r \to 0} \oint_{C_2} \mu_o \lambda^2 \mathbf{J}_\mathrm{S} \cdot d\mathbf{l}$ $\Longrightarrow$ $\lim_{r \to 0} \mathbf{J}_\mathrm{S} = \dfrac{\Phi_o}{2\pi \mu_o \lambda^2} \dfrac{1}{r} \mathbf{i}_\phi$

Normal Core of the Vortex

The current density $\lim_{r \to 0} \mathbf{J}_s = \frac{\Phi_o}{2\pi\mu_o\lambda^2}\frac{1}{r}\mathbf{i}_\phi$ diverges near the vortex center,

Which would mean that the kinetic energy of the superelectrons would also diverge. So to prevent this, below some core radius ξ the electrons become normal. This happens when the increase in kinetic energy is of the order of the gap energy. The maximum current density is then

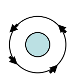
$$\mathbf{J}_s^{\text{max}} = \frac{\Phi_o}{2\pi\mu_o\lambda^2}\frac{1}{\xi}\mathbf{i}_\phi \quad \Rightarrow \quad \mathbf{v}_s^{\text{max}} = \frac{\hbar}{m^\star}\frac{1}{\xi}\mathbf{i}_\phi$$

In the absence of any current flux, the superelectrons have zero net velocity but have a speed of the fermi velocity, v_F. Hence the kinetic energy with currents is

$$\mathcal{E}_{\text{kin}}^0 = \frac{1}{2}m^\star v_F^2 = \frac{1}{2}m^\star\left(v_{F,x}^2 + v_{F,y}^2 + v_{F,z}^2\right)$$

Coherence Length ξ

The energy of a superelectron at the core is



$$\mathcal{E}_{\text{kin}}^1 = \frac{1}{2}m^\star\left[v_{F,x}^2 + \left(v_{F,y} + v_{s,\phi}^{\text{max}}\right)^2 + v_{F,z}^2\right]$$

The difference in energy, is to first order in the change in velocity,

$$\delta\mathcal{E} \approx m^\star v_{F,y}\, v_{s,\phi}^{\text{max}} \approx \Delta$$

With $\mathbf{v}_s^{\text{max}} = \frac{\hbar}{m^\star}\frac{1}{\xi}\mathbf{i}_\phi$ this gives $\xi \approx \frac{\hbar v_F}{2\Delta}$

The full BCS theory gives the *coherence length* as $\xi_o = \frac{\hbar v_F}{\pi\Delta_o}$

Therefore the maximum current density, known as the *depairing current density*, is

$$J_{\text{depair}} \approx \frac{\Phi_o}{2\pi\mu_o\lambda^2\xi}$$

Temperature Dependence

Both the coherence length and the penetration depth diverge at T_C

$$\lim_{T\to T_c} \xi(T) = \frac{\xi(0)}{\sqrt{1-(T/T_c)}} \qquad \lim_{T\to T_c} \lambda(T) = \frac{\lambda(0)}{\sqrt{1-(T/T_c)}}$$

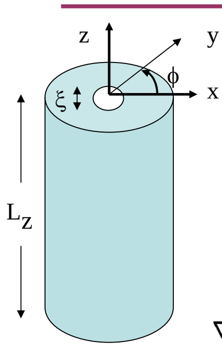
But there ratio, the Ginzburg-Landau parameter is independent of temperature near T_C

$$\kappa \equiv \frac{\lambda}{\xi}$$

$\kappa < 1/\sqrt{2}$ Type I superconductor Al, Nb

$\kappa > 1/\sqrt{2}$ Type II superconductor Nb, Most magnet materials $\kappa \gg 1$

Vortex in a Cylinder



London's Equations hold in the superconductor

$$\nabla \times (\Lambda \mathbf{J}_S) = -\mathbf{B}$$

With Ampere's Law gives

$$\nabla^2 \mathbf{B}(\mathbf{r}) - \frac{1}{\lambda^2}\mathbf{B}(\mathbf{r}) = 0 \qquad \text{for } r \geq \xi$$

Because **B** is in the z-direction, this becomes a scalar Helmholtz Equation

$$\nabla^2 B_z - \frac{1}{\lambda^2} B_z = 0 \qquad \text{for } r \geq \xi$$

$$\begin{aligned} B_z(r,\phi) \;=\; & \sum_{m=0}^{\infty} K_m\left(\frac{r}{\lambda}\right)\left(C_m \cos m\phi + C'_m \sin m\phi\right) \\ & + \sum_{m=0}^{\infty} I_m\left(\frac{r}{\lambda}\right)\left(D_m \cos m\phi + D'_m \sin m\phi\right) \end{aligned}$$

Please see: Figure 6.4, page 269, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

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Vortex in a cylinder

Which as a solution for an azimuthally symmetric field

$$B_z(r) = \begin{cases} C_0 K_0\left(\frac{r}{\lambda}\right) & \text{for } r \geq \xi \\ C_0 K_0\left(\frac{\xi}{\lambda}\right) & \text{for } r < \xi \end{cases}$$

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Please see: Figure 6.5, page 271, from Orlando, T., and K. Delin. Foundations of Applied Superconductivity. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

C_0 is found from flux quantization around the core,

$$C_0 = \frac{\Phi_o}{2\pi\lambda^2}\left[\frac{1}{2}\frac{\xi^2}{\lambda^2}K_0\left(\frac{\xi}{\lambda}\right) + \frac{\xi}{\lambda}K_1\left(\frac{\xi}{\lambda}\right)\right]^{-1}$$

Which for $\kappa \gg 1$

$$C_0 = \frac{\Phi_o}{2\pi\lambda^2}$$

Vortex in a cylinder κ >> 1

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \frac{\Phi_o}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right) \mathbf{i}_z & \text{for } r \geq \xi \\ \frac{\Phi_o}{2\pi\lambda^2} K_0\left(\frac{\xi}{\lambda}\right) \mathbf{i}_z & \text{for } r < \xi \end{cases}$$

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Please see: Figure 6.5, page 271, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

$$\mathbf{J}_s(\mathbf{r}) = \begin{cases} \frac{\Phi_o}{2\pi\mu_o\lambda^3} K_1\left(\frac{r}{\lambda}\right) \mathbf{i}_\phi & \text{for } r \geq \xi \\ 0 & \text{for } r < \xi \end{cases}$$

Energy of a single Vortex

The Electromagnetic energy in the superconducting region for a vortex is

$$W_s = \frac{1}{2\mu_o} \int_{V_s} \left[\mathbf{B}^2 + \mu_o \mathbf{J}_\mathsf{S} \cdot (\Lambda \mathbf{J}_\mathsf{S}) \right] dv$$

This gives the *energy per unit length* of the vortex as

$$\mathcal{E}_V = \frac{\Phi_o^2}{4\pi\mu_o\lambda^2} K_0\left(\frac{\xi}{\lambda}\right)$$

In the high κ limit this is

$$\lim_{\lambda \gg \xi} \mathcal{E}_V = \frac{\Phi_o^2}{4\pi\mu_o\lambda^2} \ln\left(\frac{\lambda}{\xi}\right)$$

Modified London Equation κ >> λ/ξ

Given that one is most concerned with the high κ limit, one approximates the core of the vortex ξ as a delta function which satisfies the fluxoid quantization condition. This is known as the ***Modified London Equation***:

$$\nabla \times (\Lambda \mathbf{J}_\mathsf{S}) + \mathbf{B} = \mathbf{V}(\mathbf{r})$$

The vorticity is given by delta function along the direction of the core of the vortex and the strength of the vortex is Φ_0

For a single vortex along the z-axis:

$$\mathbf{V}(\mathbf{r}) = \Phi_o \delta_2(\mathbf{r})\, \mathbf{i}_z$$

For multiple vortices

$$\mathbf{V}(\mathbf{r}) = \sum_p \Phi_o\, \delta_2(\mathbf{r} - \mathbf{r}_p)\, \mathbf{i}_z$$

General Thermodynamic Concepts

First Law of Thermodynamics: *conservation of energy*

$$\underbrace{dU}_{\text{Internal energy}} = \underbrace{dQ}_{\text{Heat in}} + \underbrace{dW}_{\text{E\&M energy stored}} - \underbrace{f_\eta d\eta}_{\text{work done by the system}}$$

W: Electromagnetic Energy

Normal region of Volume V_n

$$W_n = \int_{V_n} \frac{1}{2\mu_o} \mathbf{B}^2 \, dv$$

Superconducting region of Volume V_s

$$W_s = \frac{1}{2\mu_o} \int_{V_s} \left[\mathbf{B}^2 + \mu_o \mathbf{J}_S \cdot (\Lambda \mathbf{J}_S) \right] \, dv$$

In the absence of applied currents, in Method II, we have found that

$$dW = \int_V \mathbf{H} \cdot d\mathbf{B} \, dv$$

Moreover, for the simple geometries **H** is a constant, proportional to the applied field. For a H along a cylinder or for a slab, **H** is just the applied field. Therefore,

$$dW = \mathbf{H} \cdot d \int_V \mathbf{B} \, dv$$

Thermodynamic Fields

$$dW = \mathbf{H} \cdot d \int_V \mathbf{B}\, dv$$

$\vec{\mathcal{H}} \equiv \mathbf{H}$ thermodynamic magnetic field

$\vec{\mathcal{B}} = \frac{1}{V} \int_V \mathrm{B}$ thermodynamic flux density

$\vec{\mathcal{M}} = \frac{1}{\mu_o} \vec{\mathcal{B}} - \vec{\mathcal{H}}$ thermodynamic magnetization density

Therefore, the thermodynamic energy stored can be written simply as

$$dW = V\vec{\mathcal{H}} \cdot d\vec{\mathcal{B}}$$

Entropy and the Second Law

The *entropy S* is defined in terms of the heat delivered to a system at a temperature T

$$dS \equiv \frac{dQ}{T}$$

Second Law of Thermodynamics:

For an isolated system in equilibrium ΔS = 0

The **first law for thermodynamics** for a system in equilibrium can be written as

$$dU = T\, dS + V\vec{\mathcal{H}} \cdot d\vec{\mathcal{B}} - f_\eta d\eta$$

Then the internal energy is a function of S, *B*, and η

$$U = U(S, \mathcal{B}, \eta)$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$T, \mathcal{H}, f_\eta$ *Conjugate variables*

Concept of Reservoir and Subsystem

Because we have more control over the conjugate variables $T, \mathcal{H}, f_\eta$, we seek a rewrite the thermodynamics in terms of these controllable variables.

Isolated system = Subsystem + Reservoir

$$\Delta S_{\text{tot}} = \Delta S_A + \Delta S_R$$

The change in entropy of the reservoir is

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Please see: Figure 6.12, page 286, from Orlando, T., and K. Delin. *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

$$\Delta S_R = \frac{\Delta Q_R}{T_R} = -\frac{\Delta Q_A}{T_R}$$

Therefore, $$\Delta S_{\text{tot}} = \frac{T_R \Delta S_A - \Delta Q_A}{T_R}$$

$$\Delta S_{\text{tot}} = \frac{T_R \Delta S_A - \Delta U_A + V \vec{\mathcal{H}}_R \cdot \Delta \vec{\mathcal{B}} - f_\eta \Delta\eta}{T_R}$$

Gibbs Free Energy

The change total entropy is then

$$\Delta S_{\text{tot}} = \frac{-\Delta G_A - f_\eta \Delta\eta}{T_R} \geq 0$$

where the *Gibbs Free Energy* is defined by

$$G_A \equiv -T_R S_A + U_A - V \vec{\mathcal{H}}_R \cdot \vec{\mathcal{B}}$$

At equilibrium, the available work is just ΔG (the energy that can be freed up to do work) and the force is

$$f_\eta = -\left.\frac{\partial G}{\partial \eta}\right|_{T,\vec{\mathcal{H}}}$$

Free Energy of subsystem decreases

$$\Delta G \leq 0$$

Gibbs Free Energy and Co-energy

The Gibbs free energy is

$$G = -TS + U - V\vec{\mathcal{H}}\cdot\vec{\mathcal{B}}$$

The differential of G is

$$dG = -T\,dS - S\,dT + dU - V\,d\vec{\mathcal{H}}\cdot\vec{\mathcal{B}} - V\vec{\mathcal{H}}\cdot d\vec{\mathcal{B}}$$

and with the use of the first law $dU = T\,dS + V\vec{\mathcal{H}}\cdot d\vec{\mathcal{B}} - f_\eta d\eta$

$$dG = -S\,dT - V\,\vec{\mathcal{B}}\cdot d\vec{\mathcal{H}} - f_\eta\,d\eta$$

Therefore, the Gibbs free energy is a function of $T, \mathcal{H}, \eta$

At constant temperature and no work, then $dG|_{T,\eta} = -d\widetilde{W}$ the co-energy

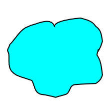
$$f_\eta = -\left.\frac{\partial G}{\partial \eta}\right|_{T,\vec{\mathcal{H}}} = \left.\frac{\partial \widetilde{W}}{\partial \eta}\right|_{T,\vec{\mathcal{H}}}$$

Note minus sign!

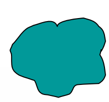
Gibbs Free Energy and Equilibruim

In Equilibrium $\Delta G = 0$

Consider the system made up of two phases 1 and 2



Phase 1, $G = G_1$



Phase 2, $G = G_2$

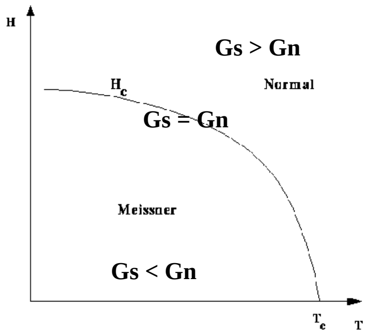
Mixed phase $G_{\text{tot}} = G_1\frac{V_1}{V} + G_2\frac{V_2}{V}$

Therefore, $G_{\text{tot}} = (G_1 - G_2)\frac{V_1}{V} + G_2$ is minimized when $G_1 = G_2$

Two phases in equilibrium with each other have the same Gibbs Free Energy

Phase Diagram and Critical Field

$\Delta G < 0$ So that G is always minimized, the system goes to the state of lowest Gibbs Free Energy. At the phase boundary, **Gs = Gn.**



At zero magnetic field in the superconducting phase

$$G_s(\vec{\mathcal{H}},0) < G_n(\vec{\mathcal{H}},0)$$

for $T < T_C$

$$G_s(0,T) - G_n(0,T) \equiv \underbrace{-\frac{1}{2}\mu_o H_c^2(T)\, V_s}_{\text{condensation enegy}}$$

The *Thermodynamic Critical Field* $H_C(T)$ is experimentally of the form

$$H_c(T) \approx H_{co}\left(1 - \left(\frac{T}{T_c}\right)^2\right) \qquad \text{for } T \le T_c$$

Critical Field for Type I

Recall that $\quad dG = -V\,\vec{\mathcal{B}} \cdot d\vec{\mathcal{H}}$

In the bulk limit in the superconducting state $B = 0$ *so that* $\quad dG_s = 0$

Likewise in the normal state $\vec{\mathcal{H}} = \mathbf{H}_{\text{app}}$ and $\vec{\mathcal{B}} = \mu_o \vec{\mathcal{H}}$ so that

$$dG_n = -V\mu_o \vec{\mathcal{H}} \cdot d\vec{\mathcal{H}}$$

Hence, we can write $\quad d\left(G_s(\vec{\mathcal{H}},T) - G_n(\vec{\mathcal{H}},T)\right) = V\,\mu_o \vec{\mathcal{H}} \cdot d\vec{\mathcal{H}}$

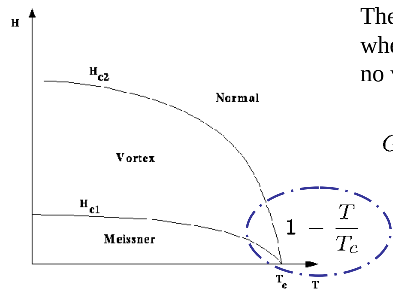
Integration of the field from 0 to H gives

$$G_s(\vec{\mathcal{H}},T) - G_n(\vec{\mathcal{H}},T) = G_s(0,T) - G_n(0,T) + \frac{1}{2} V \mu_o \vec{\mathcal{H}}^2$$

and thus

$$G_s(\mathcal{H},T) - G_n(\mathcal{H},T) = \frac{1}{2}\mu_o\left(\mathcal{H}^2 - H_c^2\right)V$$

Critical Fields for Type II



The *lower critical field* H_{C1} is the phase boundary where equilibrium between having one vortex and no vortex in the superconducting state.

$$G_s^1(\vec{\mathcal{H}}, T) = \underbrace{G_s^0(0,T)}_{G_s^0(0,\mathcal{H})} + \underbrace{W_s}_{\mathcal{E}_V L_z} - \underbrace{\vec{\mathcal{H}} \cdot \int_{V_s} \mathbf{B}\, dv}_{\Phi_o L_Z}$$

Therefore

$$H_{c1} = \frac{\mathcal{E}_V}{\Phi_o} = \frac{\Phi_o}{4\pi\mu_o\lambda^2} K_0 \underbrace{\left(\frac{\xi}{\lambda}\right)}_{\ln \lambda/\xi}$$

The *upper critical field* H_{C2} occurs when the flux density is such that the cores overlap:

$$H_{c2} = \frac{\Phi_o}{2\pi\mu_o\xi^2}$$