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High Speed Communication Circuits

Lecture 11

Noise Figure, Impact of Amplifier Nonlinearities

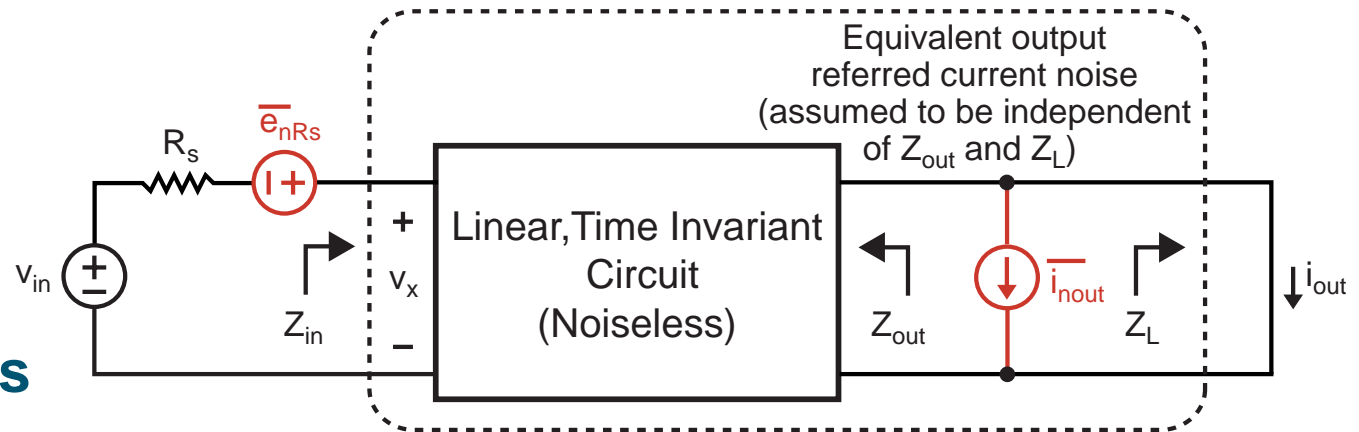
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March 10, 2005

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Noise Factor and Noise Figure (From Lec 10)



■ Definitions

$$\text{Noise Factor} = F = \frac{SNR_{in}}{SNR_{out}}$$

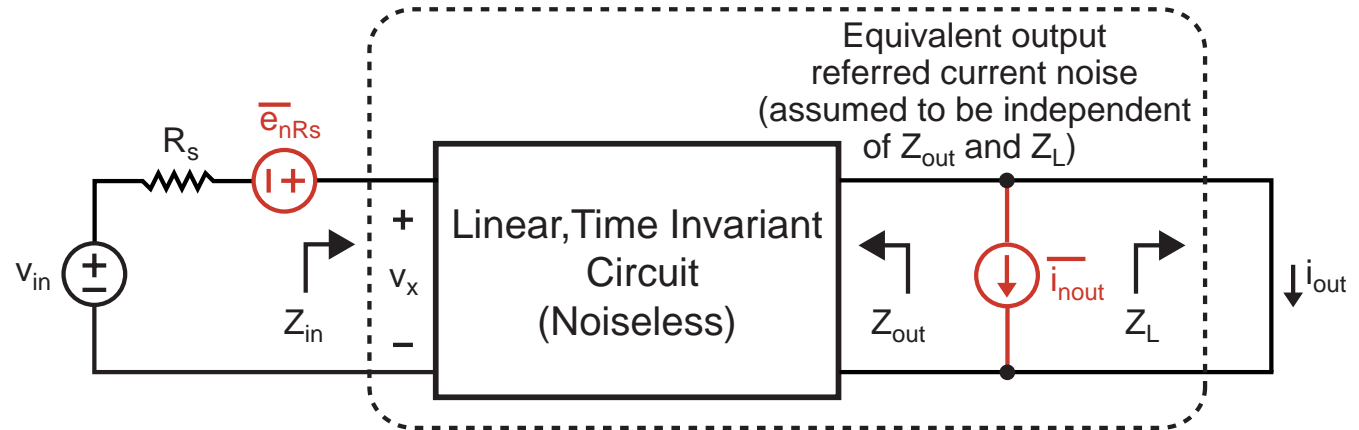
$$\text{Noise Figure} = 10 \log(\text{Noise Factor})$$

■ Calculation of SNR_{in} and SNR_{out}

$$SNR_{in} = \frac{|\alpha|^2 v_{in}^2}{|\alpha|^2 e_{nRs}^2} = \frac{v_{in}^2}{e_{nRs}^2} \quad \text{where } \alpha = \frac{Z_{in}}{R_s + Z_{in}}$$

$$SNR_{out} = \frac{|\alpha|^2 |G_m|^2 v_{in}^2}{|\alpha|^2 |G_m|^2 e_{nRs}^2 + i_{nout}^2} \quad \text{where } G_m = \frac{i_{out}}{v_x}$$

Alternative Noise Factor Expression



- From previous slide

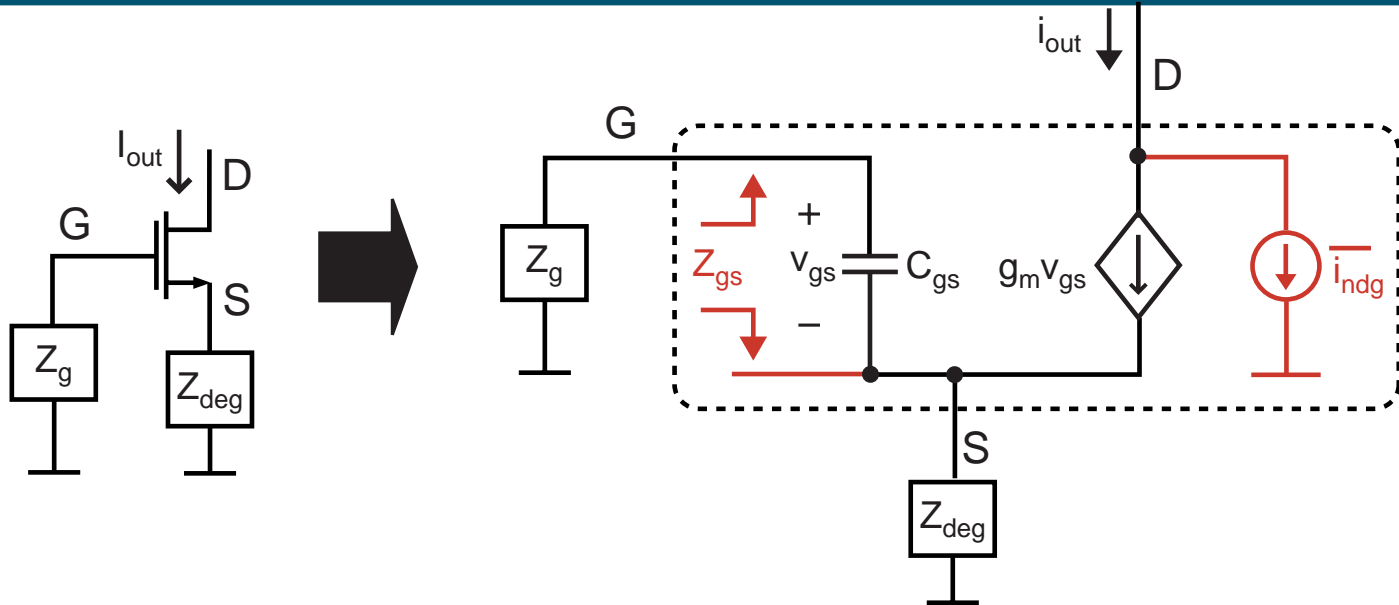
$$SNR_{in} = \frac{|\alpha|^2 v_{in}^2}{|\alpha|^2 e_{nRs}^2} = \frac{v_{in}^2}{e_{nRs}^2} \quad SNR_{out} = \frac{|\alpha|^2 |G_m|^2 v_{in}^2}{|\alpha|^2 |G_m|^2 e_{nRs}^2 + i_{nout}^2}$$

- Calculation of Noise Factor

$$F = \frac{SNR_{in}}{SNR_{out}} = \frac{|\alpha|^2 |G_m|^2 e_{nRs}^2 + i_{nout}^2}{|\alpha|^2 |G_m|^2 e_{nRs}^2}$$

$$= \frac{\text{total output noise power}}{\text{output noise due to input source}}$$

Thevenin Computation Model For Noise (from Lec 10)

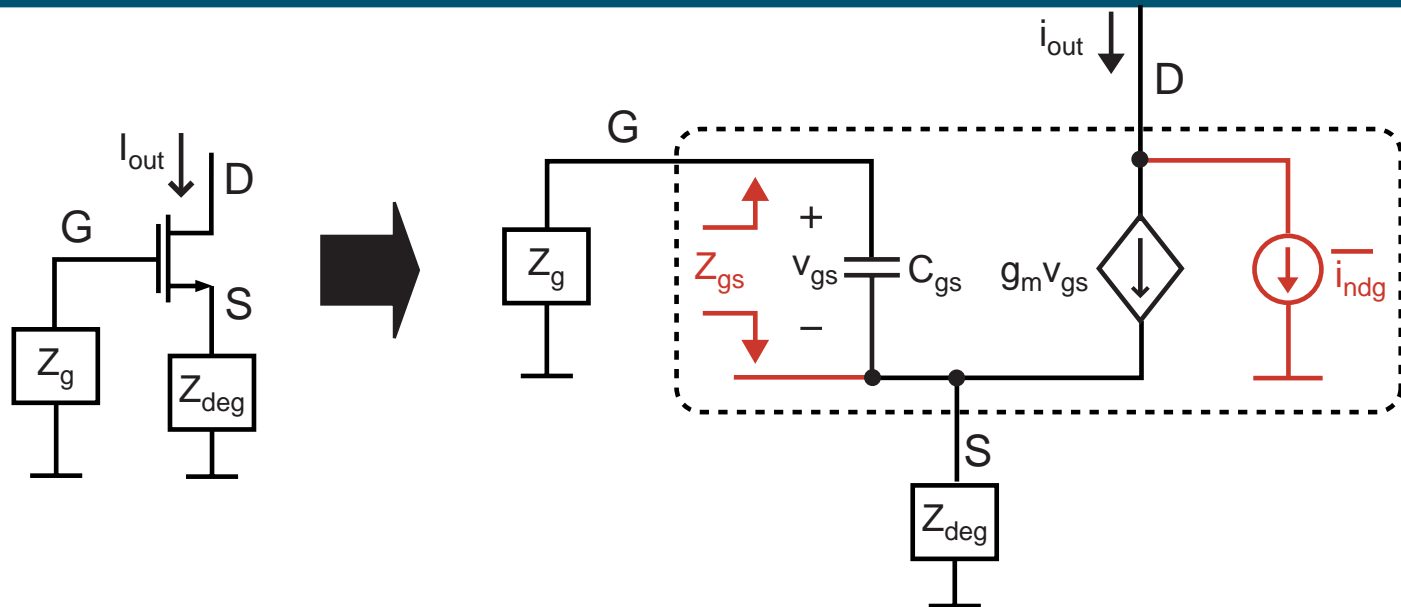


$$\frac{\overline{i_{ndg}^2}}{\Delta f} = \frac{\overline{i_{nd}^2}}{\Delta f} \left(|\eta|^2 + 2 \operatorname{Re} \{ c \chi_d \eta^* Z_{gsw} \} + \chi_d^2 |Z_{gsw}|^2 \right)$$

where: $\frac{\overline{i_{nd}^2}}{\Delta f} = 4kT\gamma g_{do}$, $\chi_d = \frac{g_m}{g_{do}} \sqrt{\frac{\delta}{5\gamma}}$, $Z_{gsw} = \omega C_{gs} Z_{gs}$

$$Z_{gs} = \frac{1}{sC_{gs}} \parallel \frac{Z_{deg} + Z_g}{1 + g_m Z_{deg}} \quad \eta = 1 - \left(\frac{g_m Z_{deg}}{Z_{deg} + Z_g} \right) Z_{gs}$$

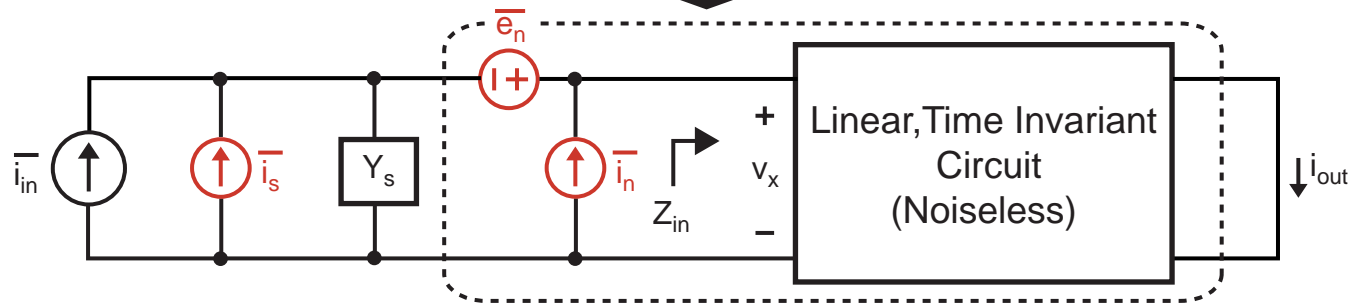
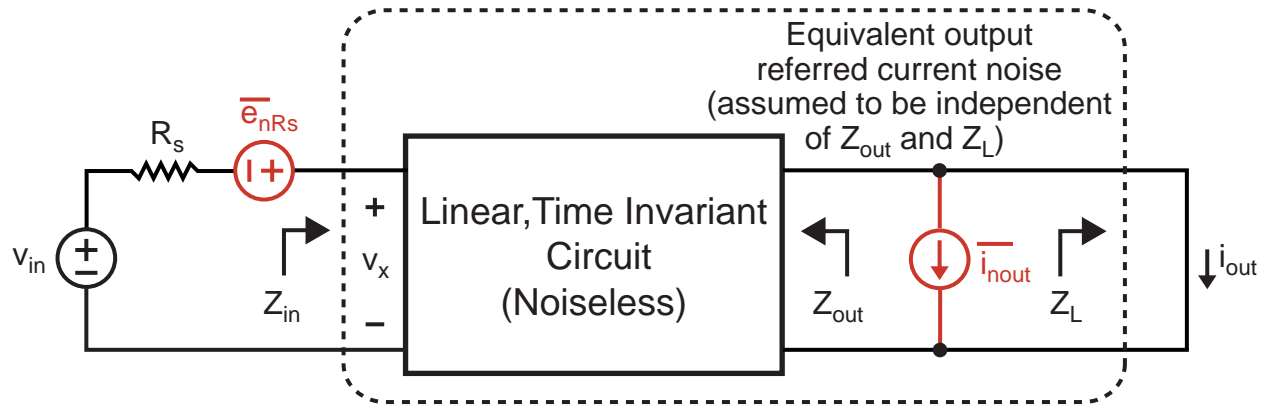
Thevenin Computation Model For Noise (from Lec 10)



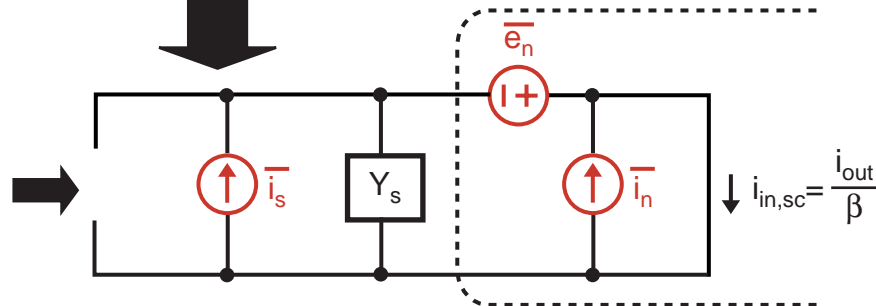
- We analyzed a non-degenerated CMOS amplifier in Lec 10
 - Broadband amplifiers: gate noise not significant
 - Narrowband amplifiers:
 - Assumed in analysis that we operated at resonance
 - Q turned out to be key design parameter

Can we simultaneously design amplifiers for optimal noise match and optimal power match?

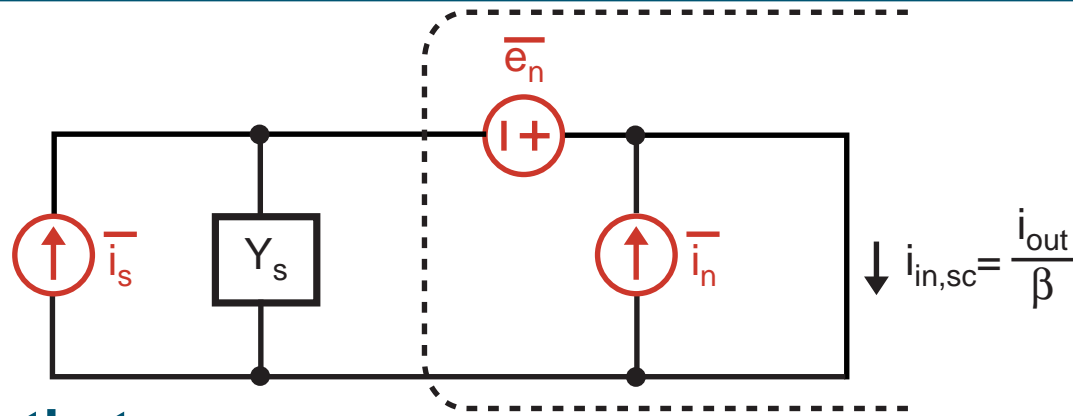
Input Referred Noise Model



Can remove the signal source since Noise Factor can be expressed as the ratio of total output noise to input noise



Input-Referred Noise Figure Expression



- We know that

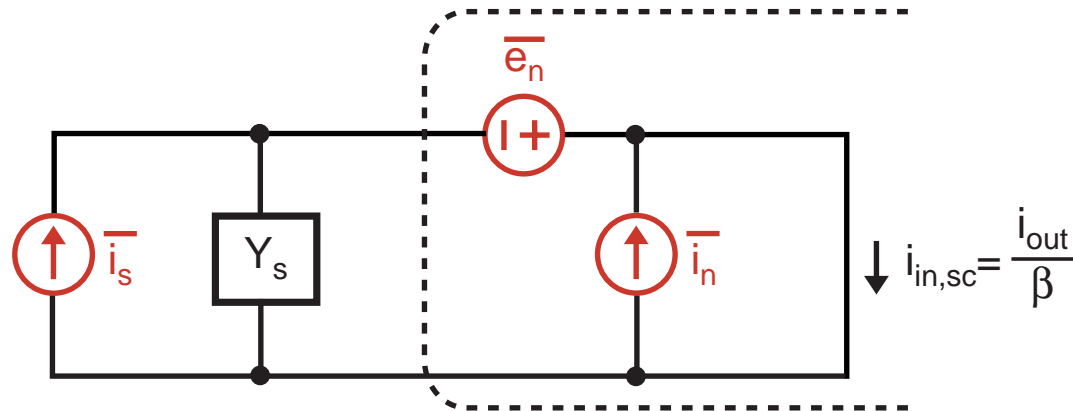
$$F = \frac{\text{total output noise power}}{\text{output noise due to input source}} = \frac{\overline{i_{out}^2(tot)}}{\overline{i_{out}^2(in)}}$$

- Let's express the above in terms of input short circuit current

$$F = \frac{|\beta|^2 \overline{i_{in,sc}^2(tot)}}{|\beta|^2 \overline{i_{in,sc}^2(in)}} = \frac{\overline{i_{in,sc}^2(tot)}}{\overline{i_{in,sc}^2(in)}}$$

$$= \frac{\text{total input short circuit current noise power}}{\text{input short circuit current noise power due to input source}}$$

Calculation of Noise Factor



- By inspection of above figure

$$F = \frac{\overline{i_{in,sc}^2(tot)}}{\overline{i_{in,sc}^2(in)}} = \frac{\overline{i_s^2} + \overline{|i_n + Y_s e_n|^2}}{\overline{i_s^2}}$$

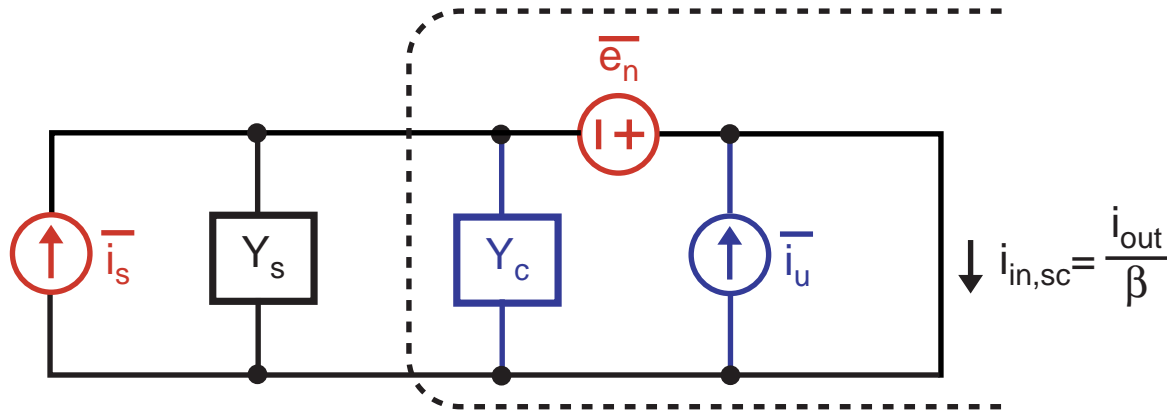
- In general, e_n and i_n will be correlated

$$i_n = i_c + i_u, \quad \text{where } i_c = Y_c e_n$$

- Y_c is called the correlation admittance

$$\Rightarrow F = \frac{\overline{i_s^2} + \overline{|i_u + (Y_c + Y_s)e_n|^2}}{\overline{i_s^2}} = 1 + \frac{\overline{i_u^2} + \overline{|Y_c + Y_s|^2 e_n^2}}{\overline{i_s^2}}$$

Noise Factor Expressed in Terms of Admittances



$$F = 1 + \frac{\bar{i}_u^2 + |Y_c + Y_s|^2 \bar{e}_n^2}{\bar{i}_s^2}$$

- We can replace voltage and current noise currents with impedances and admittances

$$R_n = \frac{\bar{e}_n^2}{4kT\Delta f}, \quad G_u = \frac{\bar{i}_u^2}{4kT\Delta f}, \quad G_s = \frac{\bar{i}_s^2}{4kT\Delta f}$$

$$F = 1 + \frac{G_u + |Y_c + Y_s|^2 R_n}{G_s}$$

Optimal Source Admittance for Minimum Noise Factor

$$F = 1 + \frac{G_u + |Y_c + Y_s|^2 R_n}{G_s}$$

- Express admittances as the sum of conductance, **G**, and susceptance, **B**

$$F = 1 + \frac{G_u + [(G_c + G_s)^2 + (B_c + B_s)^2] R_n}{G_s}$$

- Take the derivative with respect to source admittance and set to zero (to find minimum F), which yields

$$G_s = \sqrt{\frac{G_u}{R_n} + G_c^2} = G_{opt} \quad B_s = -B_c = B_{opt}$$

- Plug these values into expression above to obtain

$$F_{min} = 1 + 2R_n \left(\sqrt{\frac{G_u}{R_n} + G_c^2} + G_c \right) = 1 + 2R_n(G_{opt} + G_c)$$

Optimal Source Admittance for Minimum Noise Factor

$$F_{min} = 1 + 2R_n \left(\sqrt{\frac{G_u}{R_n} + G_c^2} + G_c \right) \quad \text{at } G_s = G_{opt}, B_s = B_{opt}$$

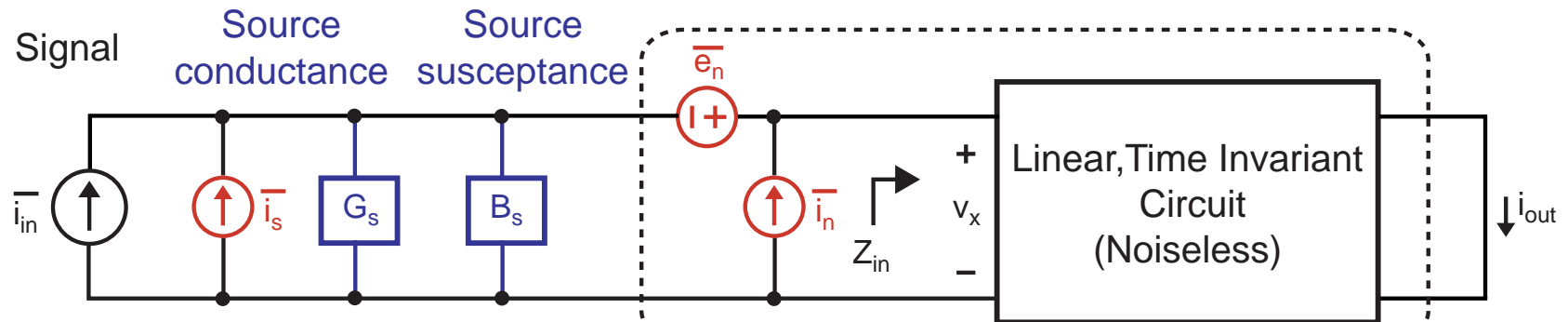
- After much algebra (see Appendix L of Gonzalez* book for derivation), we can derive

$$F = F_{min} + \frac{R_n}{G_s} \left[(G_s - G_{opt})^2 + (B_s - B_{opt})^2 \right]$$

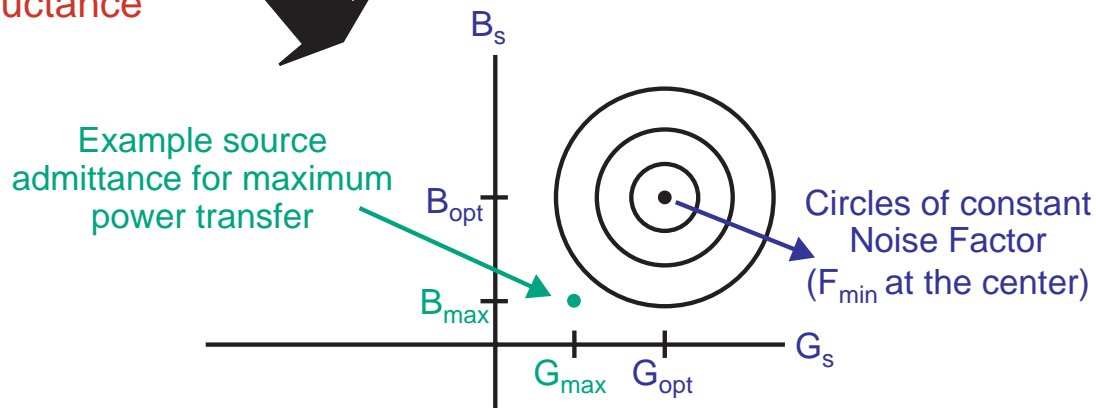
- Contours of constant noise factor are circles centered about (G_{opt}, B_{opt}) in the admittance plane
- They are also circles on a Smith Chart (see pp 299-302 of Gonzalez for derivation and examples)
- How does (G_{opt}, B_{opt}) compare to admittance achieving maximum power transfer?

*Guillermo Gonzalez, Microwave Transistor Amplifiers: Analysis and Design, Prentice Hall, 1996

Optimizing For Noise Figure versus Power Transfer



Source noise produced by source conductance



- One cannot generally achieve minimum noise figure if maximum power transfer is desired

Optimal Noise Factor for MOS Transistor Amp

- Consider the common source MOS amp (no degeneration) considered in Lecture 10
 - In Tom Lee's book (pp. 272-276 (1st ed.), pp. 364-369 (2nd ed.)), the noise impedances are derived as

$$G_c = 0 \quad B_c = wC_{gs} \left(1 - \alpha|c| \sqrt{\frac{\delta}{5\gamma}} \right) \quad (\text{corrected})$$

$$R_n = \frac{\gamma g_{d0}}{g_m^2} = \left(\frac{\gamma}{\alpha} \right) \frac{1}{g_m} \quad G_u = \frac{\delta w^2 C_{gs}^2 (1 - |c|^2)}{5g_{d0}}$$

- The optimal source admittance values to minimize noise factor are therefore

$$B_{opt} = -B_c = -wC_{gs} \left(1 - \alpha|c| \sqrt{\frac{\delta}{5\gamma}} \right)$$

$$G_{opt} = \sqrt{\frac{G_u}{R_n} + G_c^2} = \alpha w C_{gs} \sqrt{\frac{\delta}{5\gamma} (1 - |c|^2)}$$

Optimal Noise Factor for MOS Transistor Amp (Cont.)

- Optimal admittance consists of a resistor and inductor (wrong frequency behavior – broadband match fundamentally difficult)

$$B_{opt} = -B_c = -wC_{gs} \left(1 - \alpha|c| \sqrt{\frac{\delta}{5\gamma}} \right)$$
$$G_{opt} = \sqrt{\frac{G_u}{R_n} + G_c^2} = \alpha w C_{gs} \sqrt{\frac{\delta}{5\gamma} (1 - |c|^2)}$$

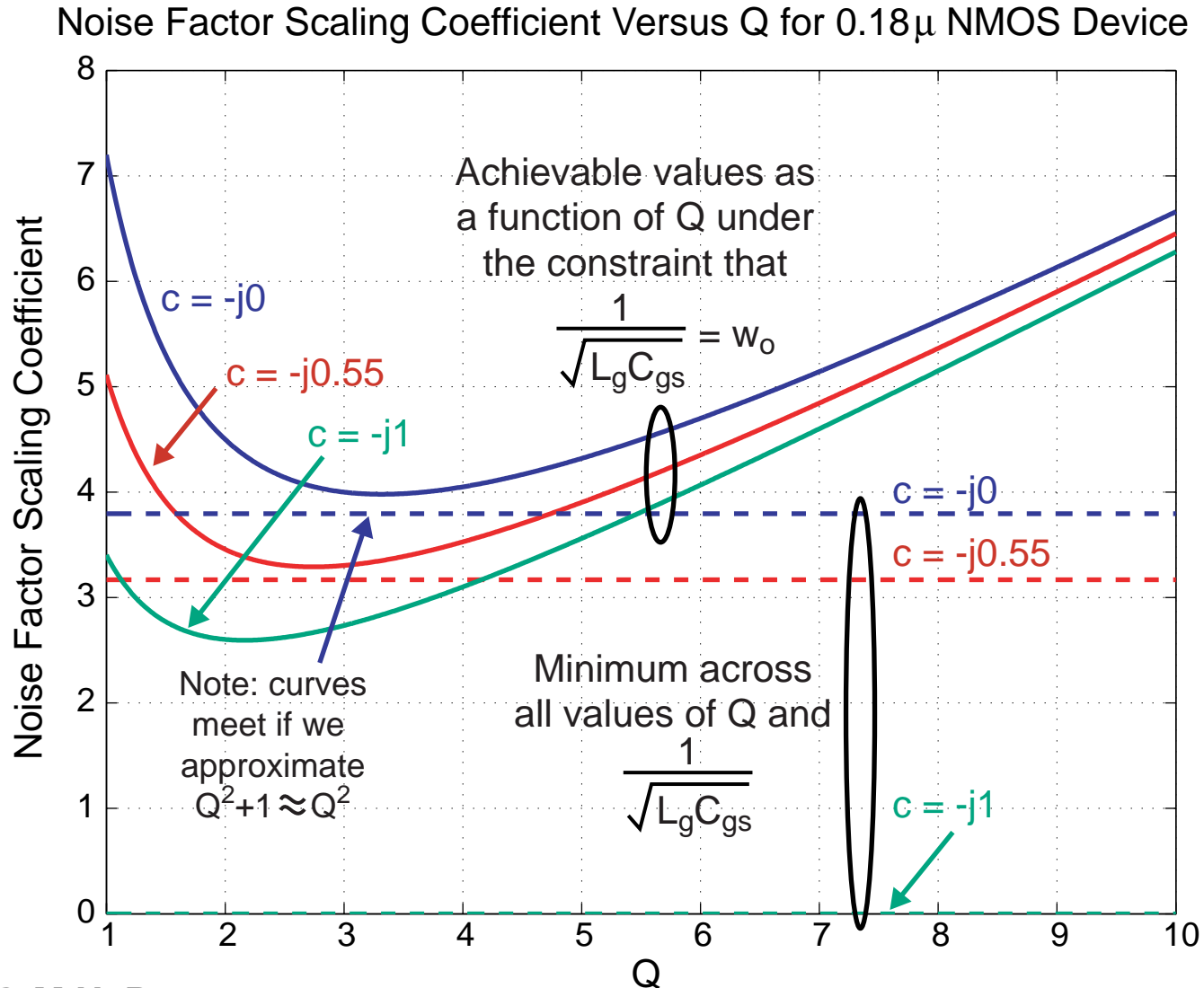
- If there is zero correlation, inductor value should be set to resonate with C_{gs} at frequency of operation

- Minimum noise figure

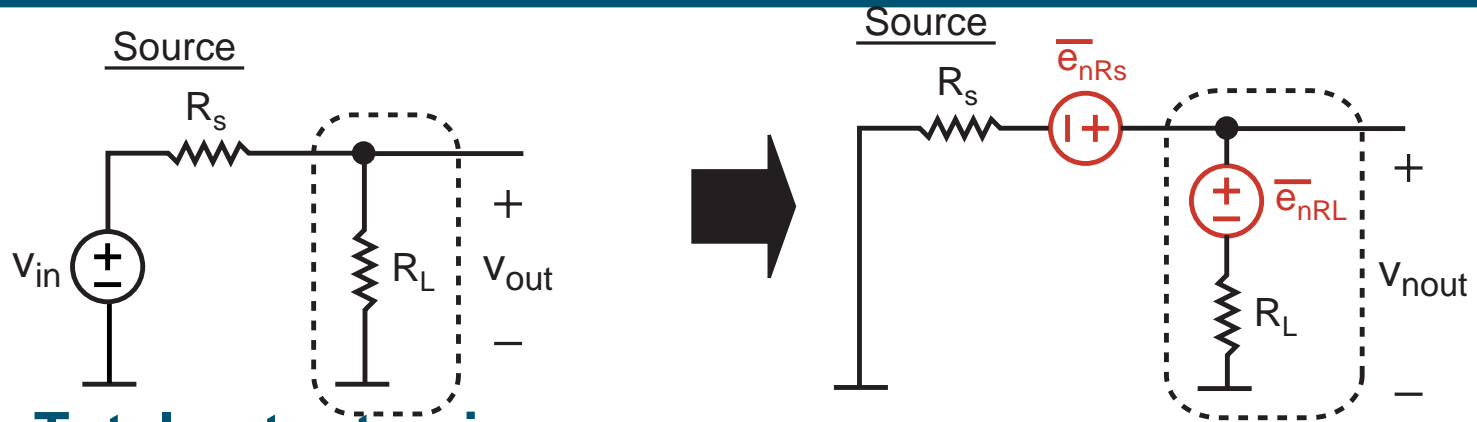
$$F_{min} = 1 + 2R_n(G_{opt} + G_c) \approx 1 + \frac{2}{\sqrt{5}} \frac{w}{w_t} \sqrt{\gamma\delta(1 - |c|^2)}$$

- Exact if one defines $w_t = g_m/C_{gs}$

Recall Noise Factor Comparison Plot From Lecture 10



Example: Noise Factor Calculation for Resistor Load



- **Total output noise**

$$\overline{v_{nout(tot)}^2} = \left(\frac{R_L}{R_s + R_L} \right)^2 \overline{e_{nRs}^2} + \left(\frac{R_s}{R_s + R_L} \right)^2 \overline{e_{nRL}^2}$$

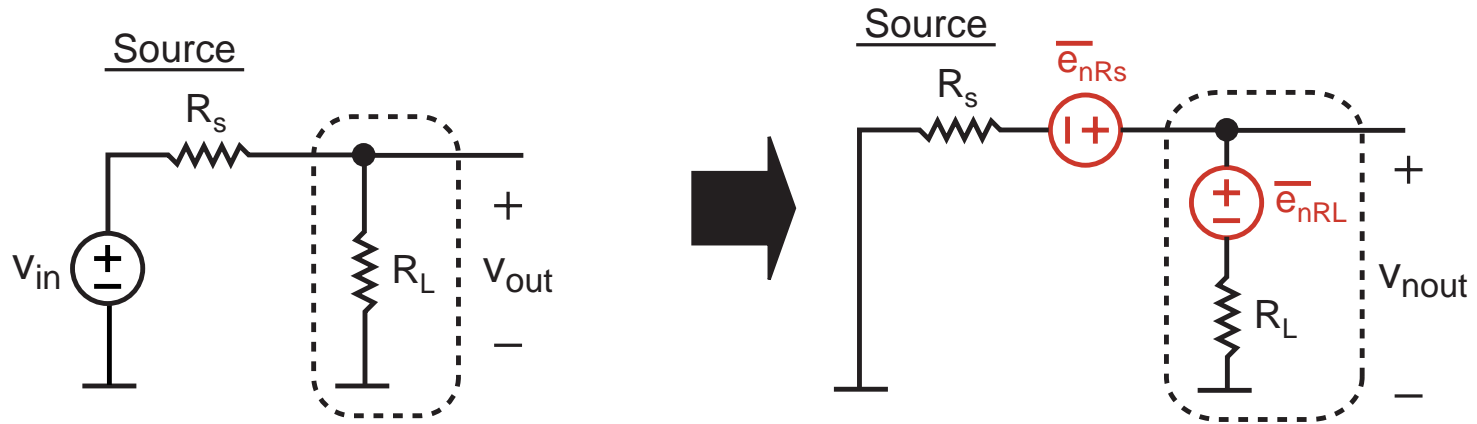
- **Total output noise due to source**

$$\overline{v_{nout(in)}^2} = \left(\frac{R_L}{R_s + R_L} \right)^2 \overline{e_{nRs}^2}$$

- **Noise Factor**

$$F = 1 + \left(\frac{R_s}{R_L} \right)^2 \frac{\overline{e_{nRL}^2}}{\overline{e_{nRs}^2}} = 1 + \left(\frac{R_s}{R_L} \right)^2 \frac{4kTR_L}{4kTR_s} = \boxed{1 + \frac{R_s}{R_L}}$$

Comparison of Noise Figure and Power Match



$$F = 1 + \frac{R_s}{R_L}$$

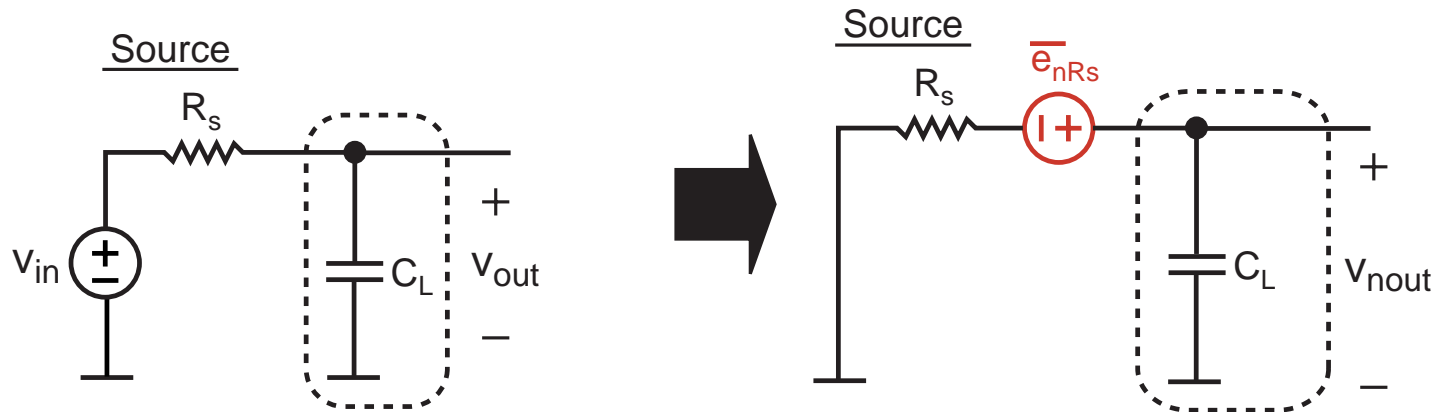
- To achieve minimum Noise Factor

$$R_L \rightarrow \infty$$

- To achieve maximum power transfer

$$R_L = R_s$$

Example: Noise Factor Calculation for Capacitor Load



- **Total output noise**

$$\overline{v_{nout}^2} = \left(\frac{1/(sC_L)}{R_s + 1/(sC_L)} \right)^2 \overline{e_{nRs}^2}$$

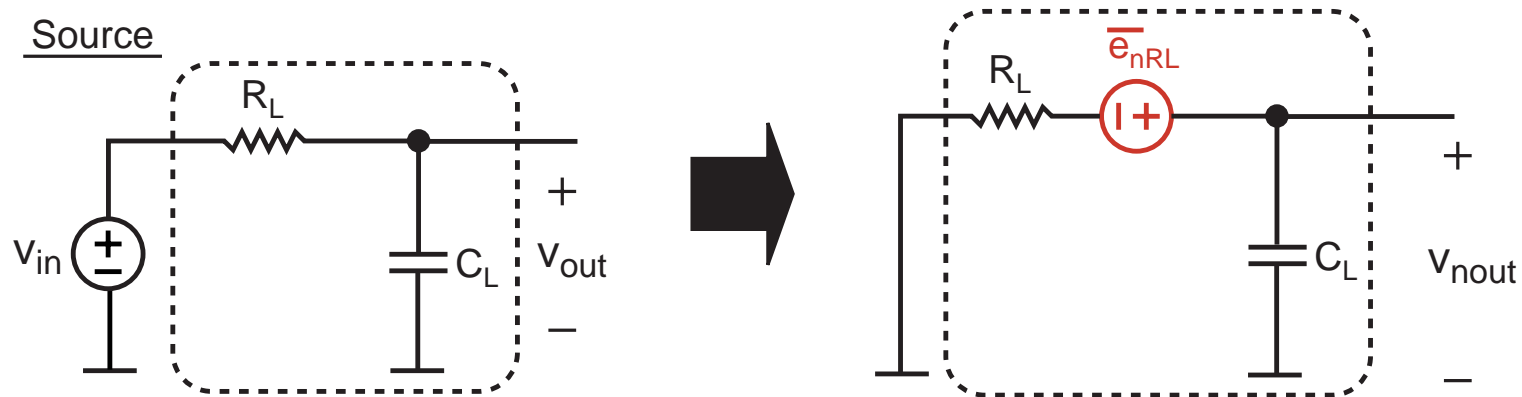
- **Total output noise due to source**

$$\overline{v_{nout}^2} = \left(\frac{1/(sC_L)}{R_s + 1/(sC_L)} \right)^2 \overline{e_{nRs}^2}$$

- **Noise Factor**

$$F = \overline{v_{nout}^2} / \overline{v_{nout}^2} = 1$$

Example: Noise Factor with Zero Source Resistance



- **Total output noise**

$$\overline{v_{nout}^2} = \left(\frac{1/(sC_L)}{R_s + 1/(sC_L)} \right)^2 \overline{e_{nRs}^2}$$

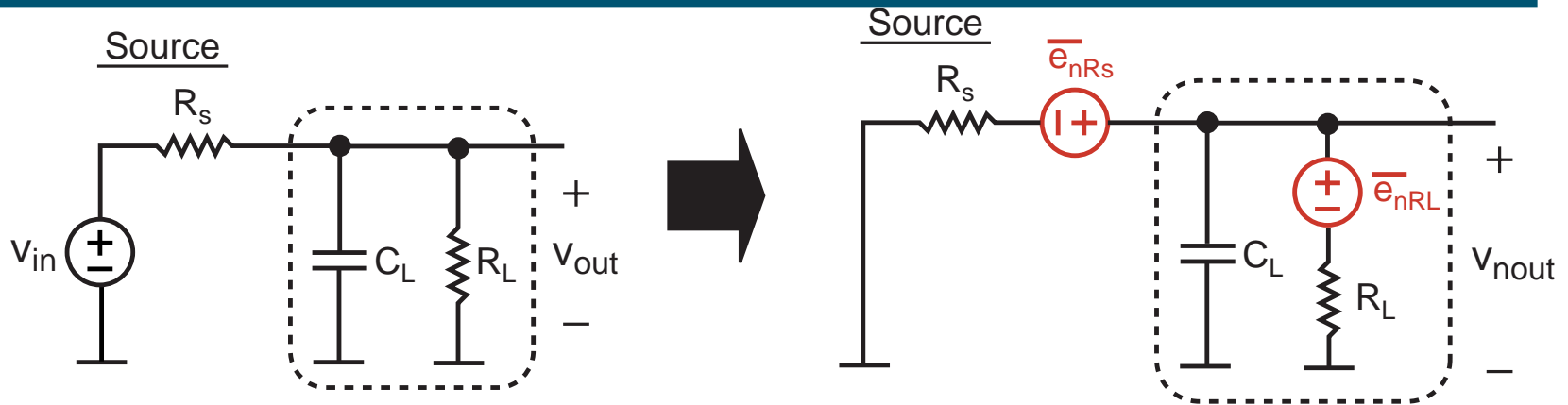
- **Total output noise due to source**

$$\overline{v_{nout}^2} = 0$$

- **Noise Factor**

$$F = \overline{v_{nout}^2} / \overline{v_{nout}^2} = \infty$$

Example: Noise Factor Calculation for RC Load



- **Total output noise**

$$= \left(\frac{1/(sC_L) \parallel R_L}{R_s + 1/(sC_L) \parallel R_L} \right)^2 \overline{e_{nRs}^2} + \left(\frac{1/(sC_L) \parallel R_s}{R_L + 1/(sC_L) \parallel R_s} \right)^2 \overline{e_{nRL}^2}$$

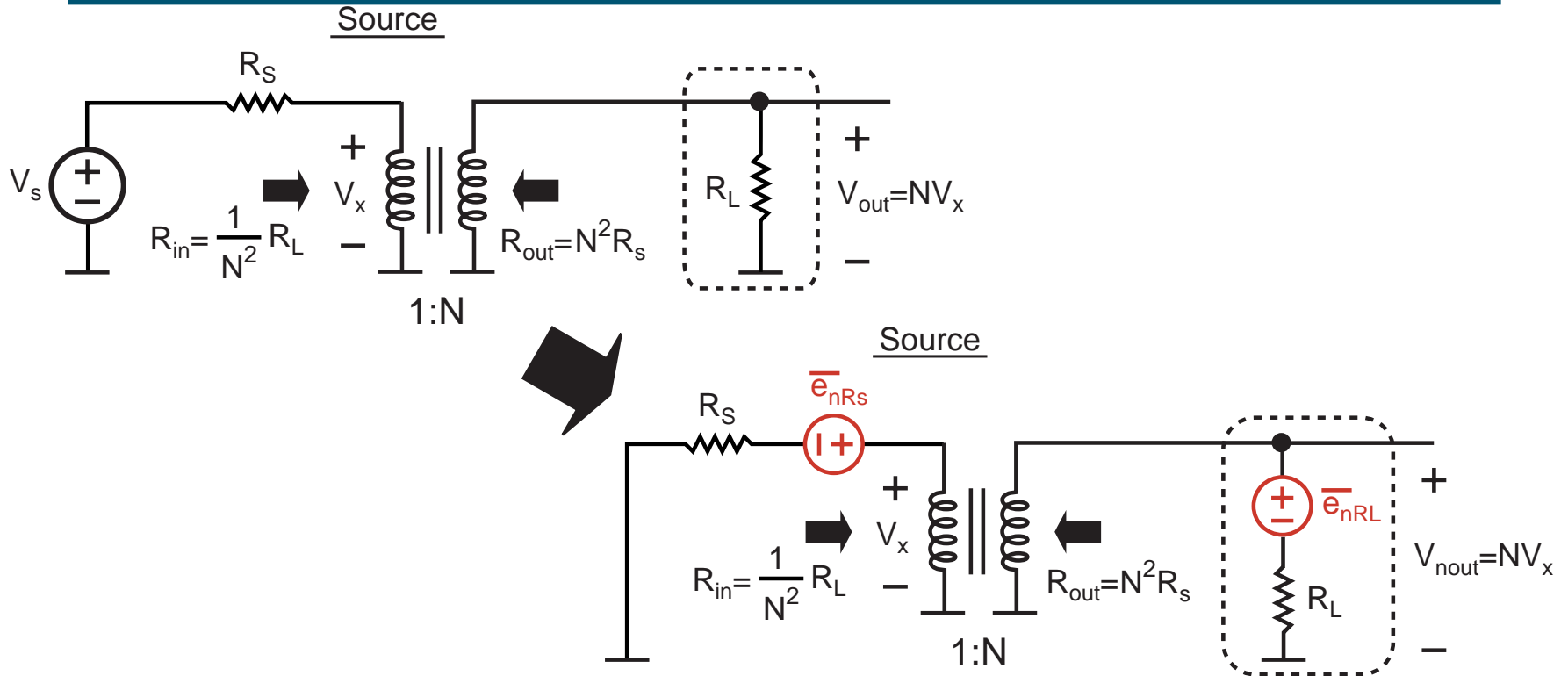
- **Total output noise due to source**

$$= \left(\frac{1/(sC_L) \parallel R_L}{R_s + 1/(sC_L) \parallel R_L} \right)^2 \overline{e_{nRs}^2}$$

- **Noise Factor**

$$F = 1 + \left(\frac{R_s}{R_L} \right)^2 \frac{\overline{e_{nRL}^2}}{\overline{e_{nRs}^2}} = 1 + \left(\frac{R_s}{R_L} \right)^2 \frac{4kTR_L}{4kTR_s} = \boxed{1 + \frac{R_s}{R_L}}$$

Example: Resistive Load with Source Transformer

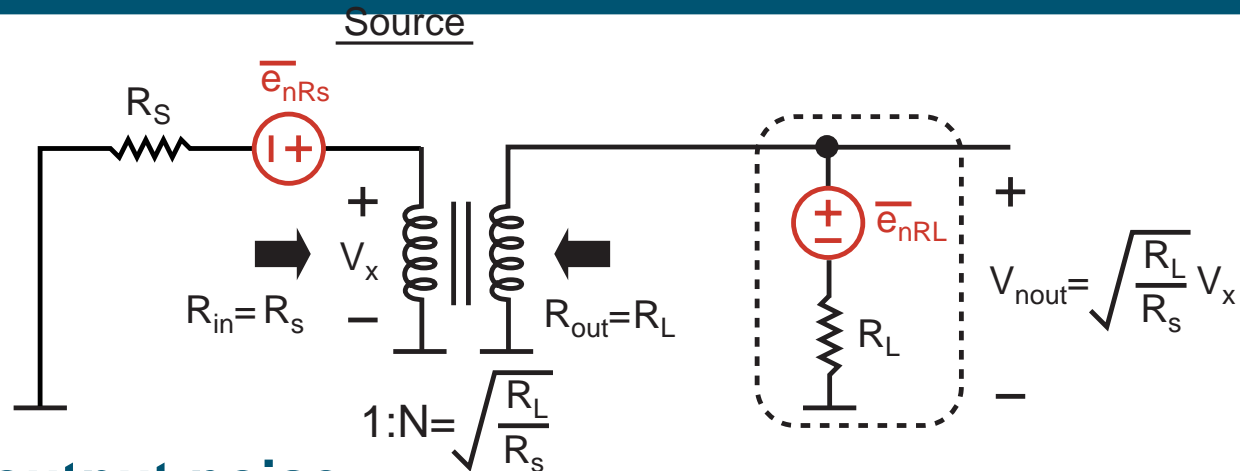


- For maximum power transfer (as derived in Lecture 3)

$$N^2 = \frac{R_L}{R_s}$$

$$\Rightarrow R_{in} = R_s, \quad R_{out} = R_L, \quad V_{out} = \sqrt{\frac{R_L}{R_s}} V_x$$

Noise Factor with Transformer Set for Max Power Transfer



- **Total output noise**

$$\overline{v_{nout}^2} = \left(\frac{R_s}{R_s + R_s} \sqrt{\frac{R_L}{R_s}} \right)^2 \overline{e_{nRs}^2} + \left(\frac{R_L}{R_L + R_L} \right)^2 \overline{e_{nRL}^2}$$

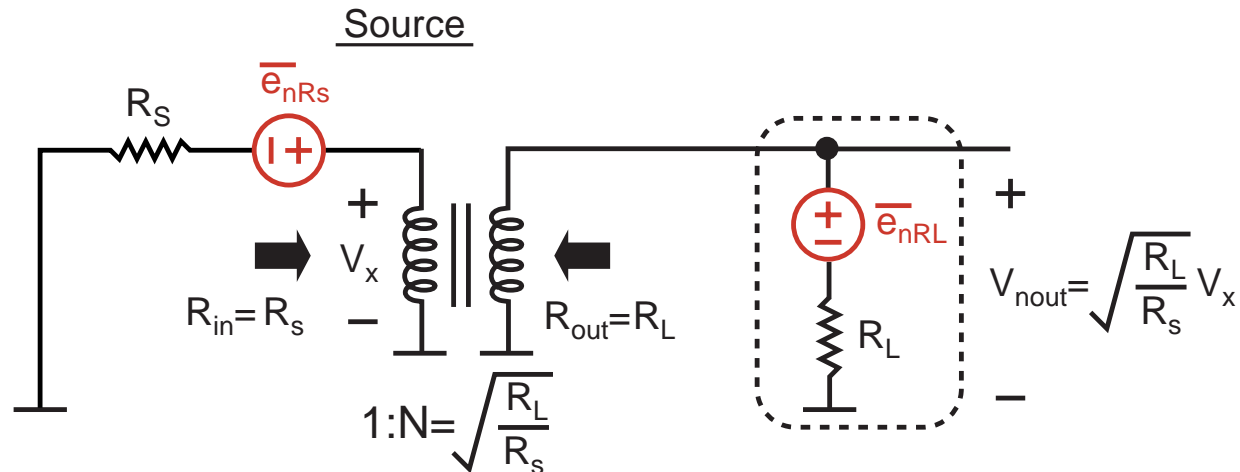
- **Total output noise due to source**

$$\overline{v_{nout}^2} = \left(\frac{R_s}{R_s + R_s} \sqrt{\frac{R_L}{R_s}} \right)^2 \overline{e_{nRs}^2}$$

- **Noise Factor**

$$F = 1 + \frac{R_s}{R_L} \frac{\overline{e_{nRL}^2}}{\overline{e_{nRs}^2}} = 1 + \left(\frac{R_s}{R_L} \right) \frac{4kTR_L}{4kTR_s} = 2 \quad (= 3dB)$$

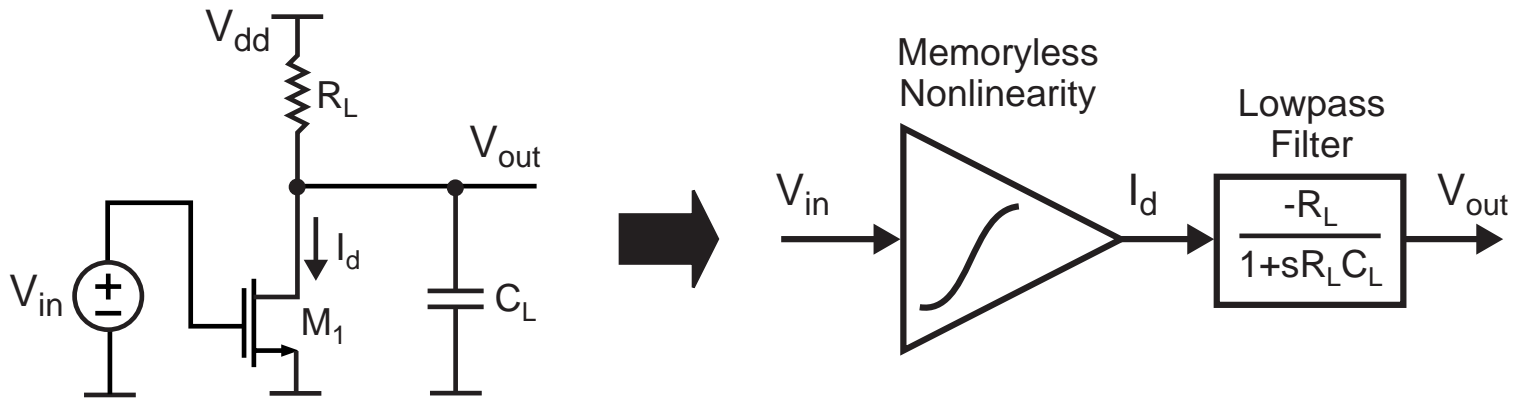
Observations



- If you need to power match to a resistive load, you must pay a 3 dB penalty in Noise Figure
 - A transformer does not alleviate this issue
- What value does a transformer provide?
 - Almost-true answer: maximizes voltage gain given the power match constraint, thereby reducing effect of noise of following amplifiers
 - Accurate answer: we need to wait until we talk about cascaded noise factor calculations

Nonlinearities in Amplifiers

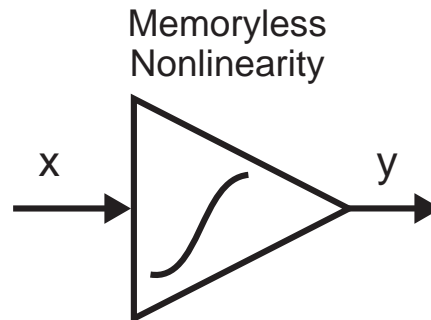
- We can generally break up an amplifier into the cascade of a memoryless nonlinearity and an input and/or output transfer function



- Impact of nonlinearities with sine wave input
 - Causes harmonic distortion (i.e., creation of harmonics)
- Impact of nonlinearities with several sine wave inputs
 - Causes harmonic distortion for each input AND intermodulation products

Analysis of Amplifier Nonlinearities

- Focus on memoryless nonlinearity block
 - The impact of filtering can be added later



- Model nonlinearity as a Taylor series expansion up to its third order term (assumes small signal variation)

$$y(t) \approx c_0 + c_1x(t) + c_2x(t)^2 + c_3x(t)^3$$

- For harmonic distortion, consider

$$x(t) = A \cos(\omega t)$$

- For intermodulation, consider

$$x(t) = A(\cos(\omega_1 t) + \cos(\omega_2 t))$$

Harmonic Distortion

$$y(t) = c_0 + c_1x(t) + c_2x(t)^2 + c_3x(t)^3$$

$$\text{where } x(t) = A \cos wt$$

- **Substitute x(t) into polynomial expression**

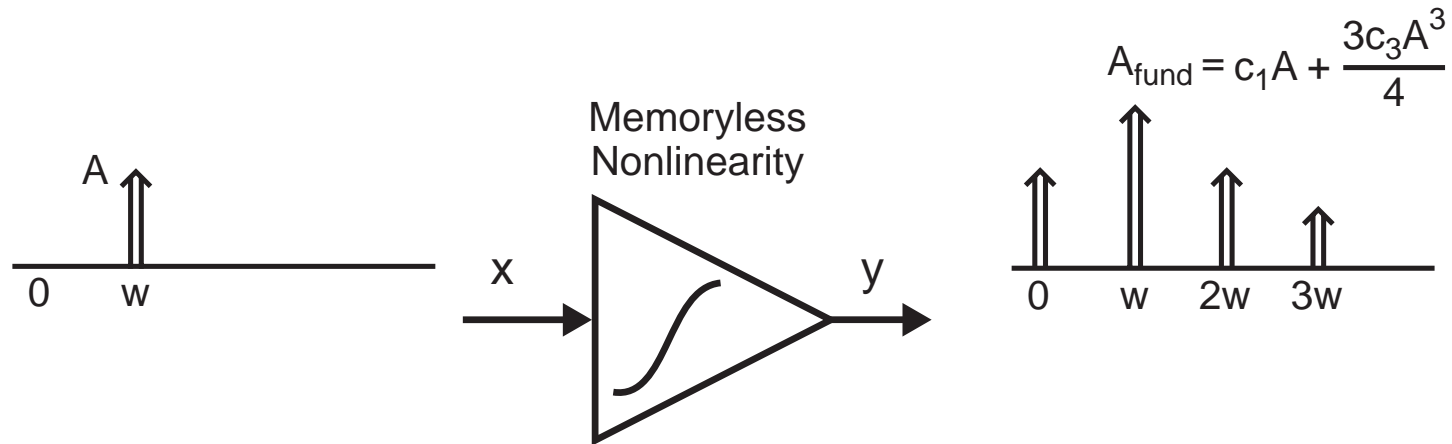
$$y(t) - c_0 = c_1A \cos wt + c_2A^2 \cos^2 wt + c_3A^3 \cos^3 wt$$

$$= c_1A \cos wt + \frac{c_2A^2}{2}(1 + \cos 2wt) + \frac{c_3A^3}{4}(3 \cos wt + \cos 3wt)$$

$$= \frac{c_2A^2}{2} + \underbrace{\left(c_1A + \frac{3c_3A^3}{4}\right) \cos wt}_{\text{Fundamental}} + \underbrace{\frac{c_2A^2}{2} \cos 2wt + \frac{c_3A^3}{4} \cos 3wt}_{\text{Harmonics}}$$

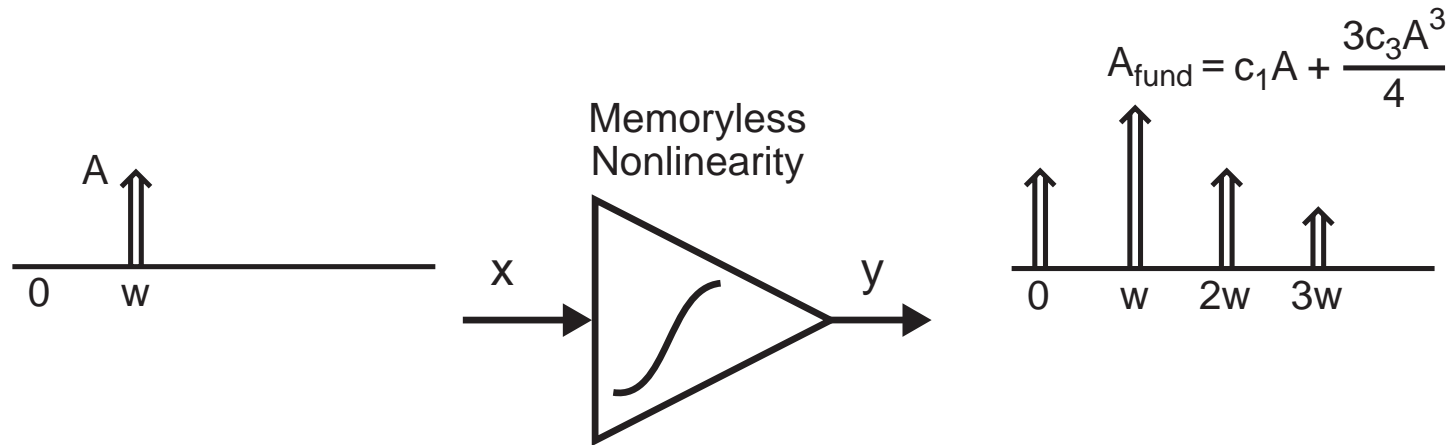
- **Notice that each harmonic term, $\cos(nwt)$, has an amplitude that grows in proportion to A^n**
 - **Very small for small A, very large for large A**

Frequency Domain View of Harmonic Distortion

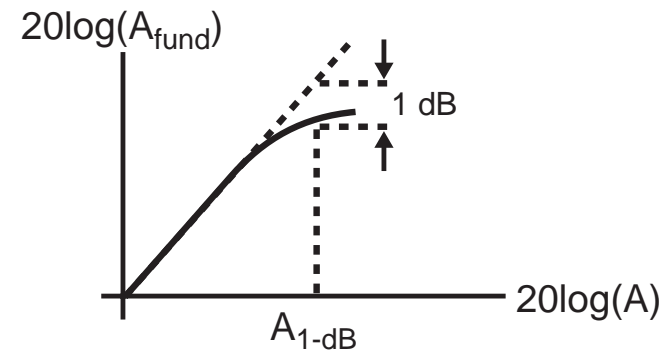


- **Harmonics cause “noise”**
 - Their impact depends highly on application
 - LNA – typically not of consequence
 - Power amp – can degrade spectral mask
 - Audio amp – depends on your listening preference!
- **Gain for fundamental component depends on input amplitude!**

1 dB Compression Point



- **Definition: input signal level such that the small-signal gain drops by 1 dB**
 - Input signal level is high!
- **Typically calculated from simulation or measurement rather than analytically**
 - Analytical model must include many more terms in Taylor series to be accurate in this context



Harmonic Products with An Input of Two Sine Waves

$$y(t) = c_0 + c_1x(t) + c_2x(t)^2 + c_3x(t)^3$$

where $x(t) = A(\cos w_1t + \cos w_2t)$

- **DC and fundamental components**

$$\left(c_0 + c_2A^2 \right) + \left((c_1A + \frac{9}{4}c_3A^3)(\cos w_1t + \cos w_2t) \right)$$

- **Second and third harmonic terms**

$$\left(\frac{c_2A^2}{2}(\cos 2w_1t + \cos 2w_2t) \right) + \left(\frac{c_3A^3}{4}(\cos 3w_1t + \cos 3w_2t) \right)$$

- **Similar result as having an input with one sine wave**
 - **But, we haven't yet considered cross terms!**

Intermodulation Products

$$y(t) = c_0 + c_1x(t) + c_2x(t)^2 + c_3x(t)^3$$

$$\text{where } x(t) = A(\cos w_1t + \cos w_2t)$$

- **Second-order intermodulation (IM2) products**

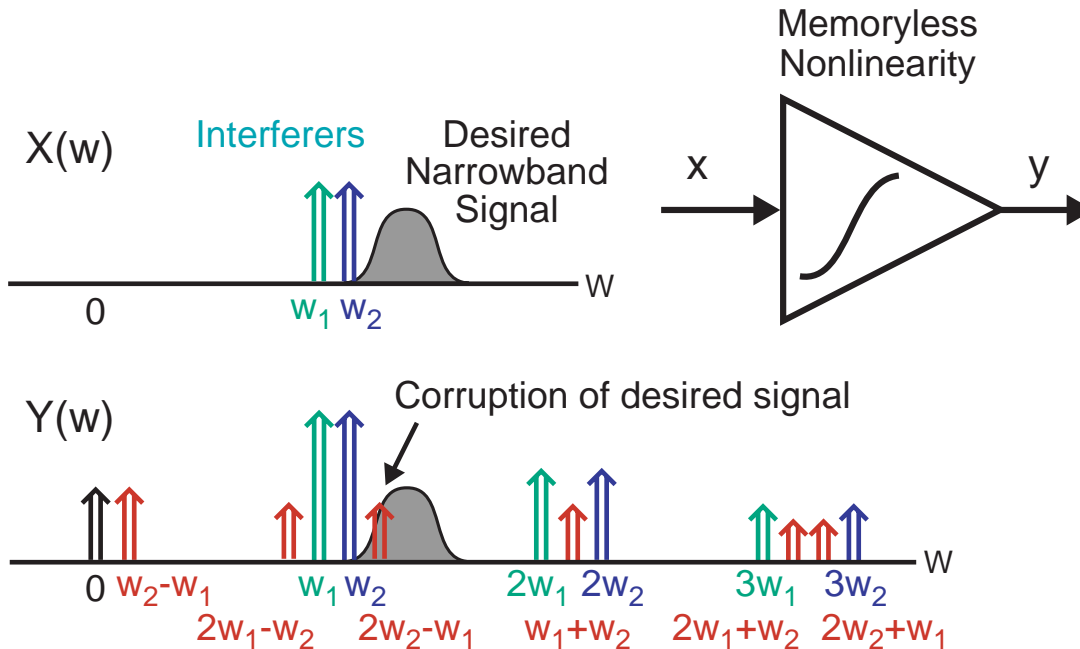
$$c_2A^2(\cos(w_1 + w_2)t + \cos(w_2 - w_1)t)$$

- **Third-order intermodulation (IM3) products**

$$\frac{3}{4}c_3A^3 \left(\cos(2w_1 + w_2)t + \cos(2w_1 - w_2)t \right. \\ \left. + \cos(2w_2 + w_1)t + \cos(2w_2 - w_1)t \right)$$

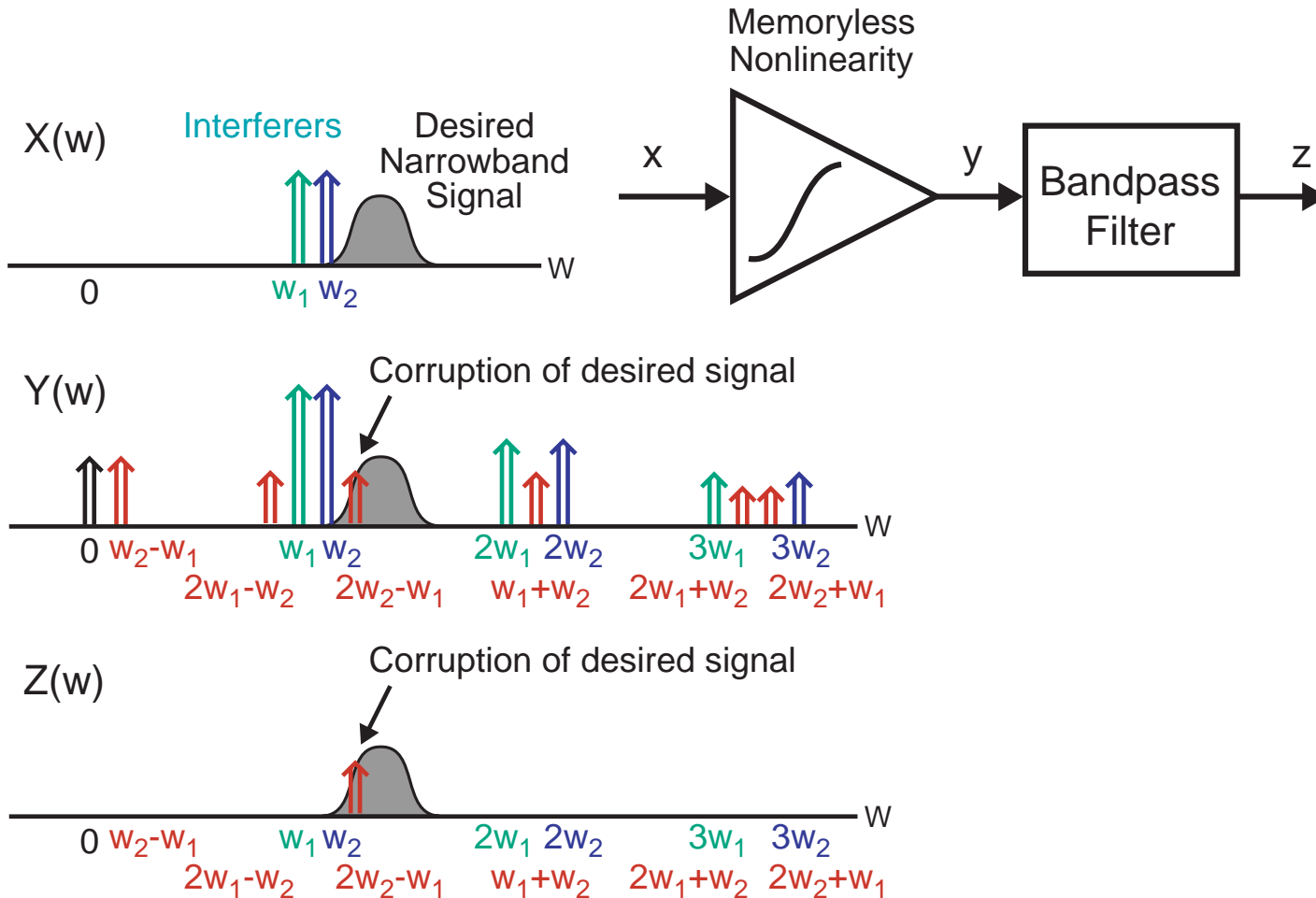
- **These are the troublesome ones for narrowband systems**

Corruption of Narrowband Signals by Interferers



- **Wireless receivers must select a desired signal that is accompanied by interferers that are often much larger**
 - LNA nonlinearity causes the creation of harmonic and intermodulation products
 - Must remove interference and its products to retrieve desired signal

Use Filtering to Remove Undesired Interference



- Ineffective for IM3 term that falls in the desired signal frequency band

Characterization of Intermodulation

- **Magnitude of third order products is set by c_3 and input signal amplitude (for small A)**

$$\frac{3}{4}c_3A^3 \left(\cos(2w_1 + w_2)t + \cos(2w_1 - w_2)t \right. \\ \left. + \cos(2w_2 + w_1)t + \cos(2w_2 - w_1)t \right)$$

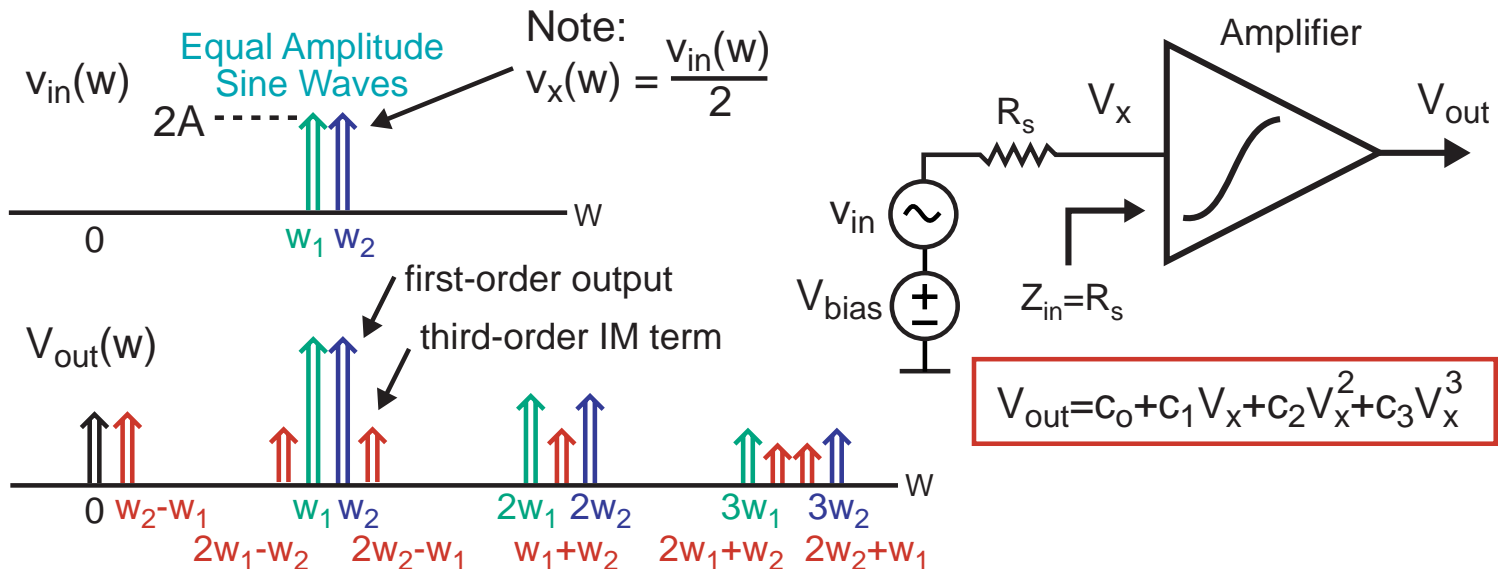
- **Magnitude of first order term is set by c_1 and A (for small A)**

$$\left(c_1A + \frac{9}{4}c_3A^3 \right) (\cos w_1t + \cos w_2t) \approx c_1A (\cos w_1t + \cos w_2t)$$

- **Relative impact of intermodulation products can be calculated once we know A and the ratio of c_3 to c_1**
 - **Problem: it's often hard to extract the polynomial coefficients through direct DC measurements**
 - **Need an indirect way to measure the ratio of c_3 to c_1**

Two Tone Test

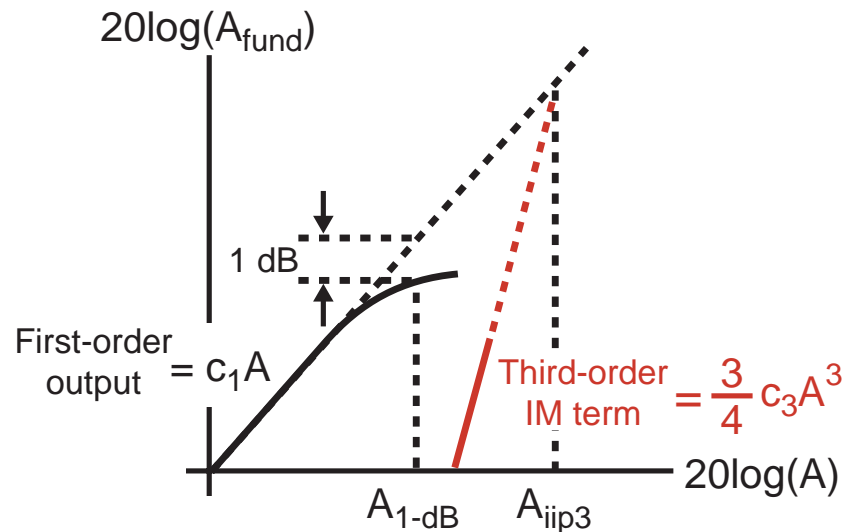
- Input the sum of two equal amplitude sine waves into the amplifier (assume Z_{in} of amplifier = R_s of source)



- On a spectrum analyzer, measure first order and third order terms as A is varied (A must remain small)
 - First order term will increase linearly
 - Third order IM term will increase as the cube of A

Input-Referred Third Order Intercept Point (IIP3)

- Plot the results of the two-tone test over a range of A (where A remains small) on a log scale (i.e., dB)
 - Extrapolate the results to find the intersection of the first and third order terms



- IIP3 defined as the input power at which the extrapolated lines intersect (higher value is better)
 - Note that IIP3 is a small signal parameter based on extrapolation, in contrast to the 1-dB compression point

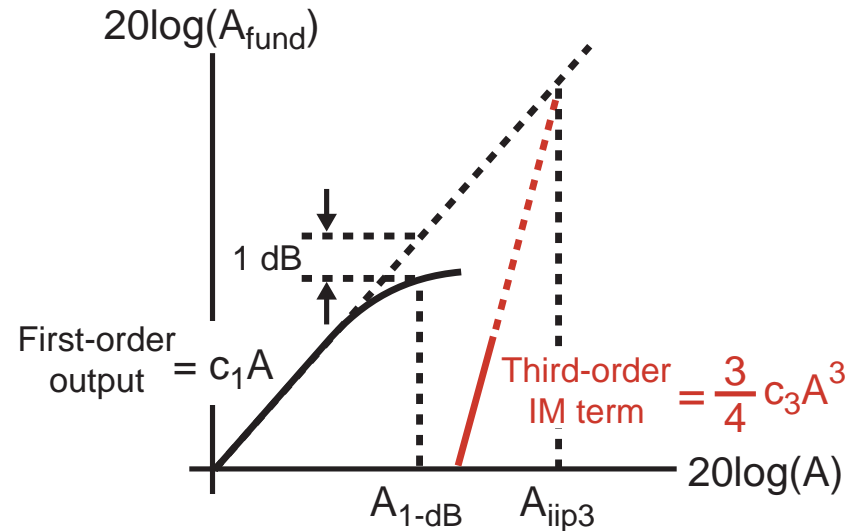
Relationship between IIP3, c_1 and c_3

- Intersection point

$$|c_1 A| = \left| \frac{3}{4} c_3 A^3 \right|$$

- Solve for A (gives A_{iip3})

$$\Rightarrow A^2 = \frac{4}{3} \left| \frac{c_1}{c_3} \right| (V_p^2)$$



- Note that A corresponds to the peak value of the two cosine waves coming into the amplifier input node (V_x)
 - Would like to instead like to express IIP3 in terms of power

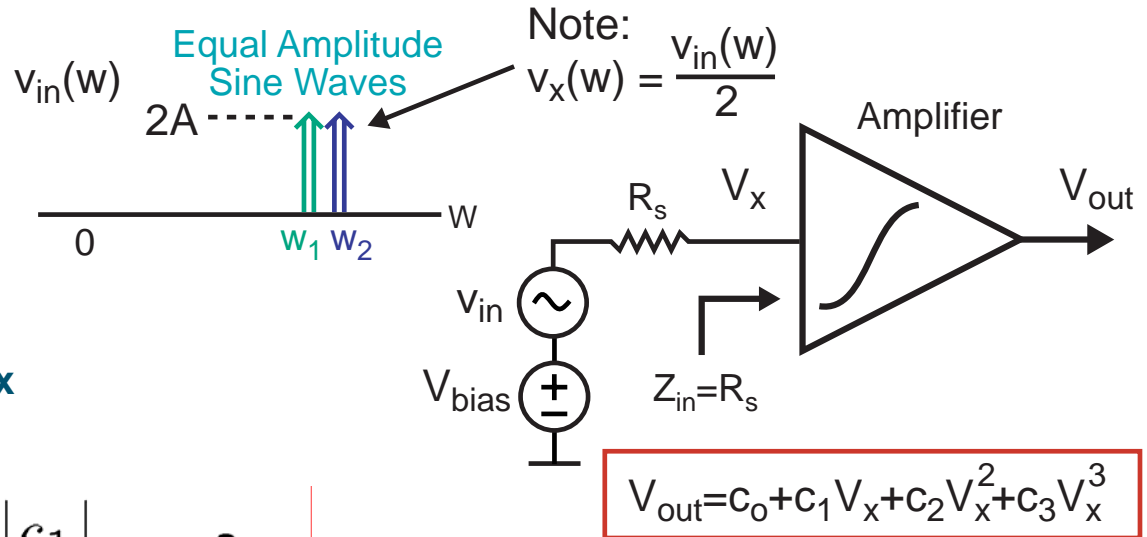
IIP3 Expressed in Terms of Power at Source

- IIP3 referenced to V_x (peak voltage)

$$A^2 = \frac{4}{3} \left| \frac{c_1}{c_3} \right| (V_p^2)$$

- IIP3 referenced to V_x (rms voltage)

$$A_{rms}^2 = \left(\frac{A}{\sqrt{2}} \right)^2 = \frac{2}{3} \left| \frac{c_1}{c_3} \right| (V_{rms}^2)$$



- Power across $Z_{in} = R_s$

$$\frac{A_{rms}^2}{R_s} = \frac{2}{3} \left| \frac{c_1}{c_3} \right| \frac{1}{R_s} \text{ (Watts)}$$

- Note: Power from v_{in}

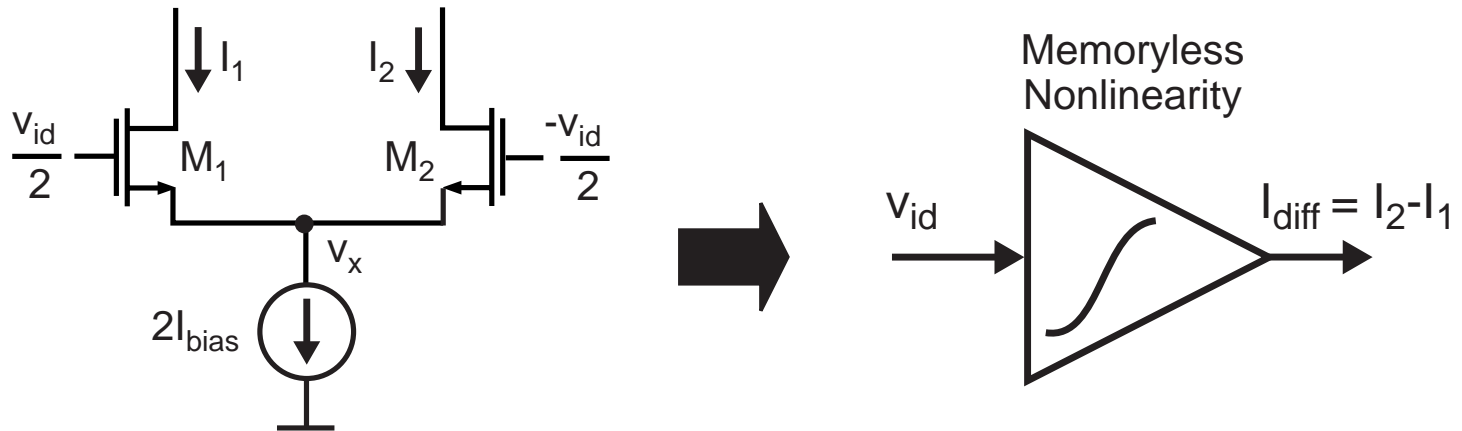
$$2 \frac{A_{rms}^2}{R_s} = \frac{4}{3} \left| \frac{c_1}{c_3} \right| \frac{1}{R_s} \text{ (Watts)}$$

$$\Rightarrow IIP3(dBm) = 10 \log \left((10^3) \frac{2}{3} \left| \frac{c_1}{c_3} \right| \frac{1}{R_s} \right) \text{ (dBm)}$$

IIP3 as a Benchmark Specification

- **Since IIP3 is a convenient parameter to describe the level of third order nonlinearity in an amplifier, it is often quoted as a benchmark spec**
- **Measurement of IIP3 on a discrete amplifier would be done using the two-tone method described earlier**
 - This is rarely done on integrated amplifiers due to poor access to the key nodes
 - Instead, for a radio receiver for instance, one would simply put in interferers and see how the receiver does
 - Note: performance in the presence of interferers is not just a function of the amplifier nonlinearity
- **Calculation of IIP3 is most easily done using a simulator such as Hspice or Spectre**
 - Two-tone method is *theoretically* not necessary – simply curve fit to a third order polynomial

Impact of Differential Amplifiers on Nonlinearity



- Assume v_x is approximately incremental ground

$$I_{diff} = c_0 + c_1 \frac{v_{id}}{2} + c_2 \left(\frac{v_{id}}{2} \right)^2 + c_3 \left(\frac{v_{id}}{2} \right)^3 - \left(c_0 + c_1 \frac{-v_{id}}{2} + c_2 \left(\frac{-v_{id}}{2} \right)^2 + c_3 \left(\frac{-v_{id}}{2} \right)^3 \right)$$

$$\Rightarrow I_{diff} = c_1 v_{id} + \frac{c_3}{4} v_{id}^3$$

- Second order term removed and IIP3 increased!