MIT OPENCOURSEWARE
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.776
High Speed Communication Circuits and Systems
Lecture 5
Generalized Reflection Coefficient, Smith Chart

Massachusetts Institute of Technology

February 15, 2005

Reflection Coefficient

- **We defined, at the load** $\Gamma_L = \frac{V_r}{V_i}$

- **Load and characteristic impedances were related**

$$Z_o \left(\frac{1 + \Gamma_L}{1 - \Gamma_L} \right) = Z_L$$

- **Alternately**

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$$

Can we find reflection coefficient at locations other than the load location?: Generalized reflection coefficient

Voltage and Current Waveforms

- In Lecture 3, we found that in sinusoidal steady-state in a transmission line,

$$V(z,t) = V_o e^{j(wt \pm kz)}$$

where the – sign is for the wave traveling in the +z direction,a nd the + sign is for the wave traveling in the –z direction.

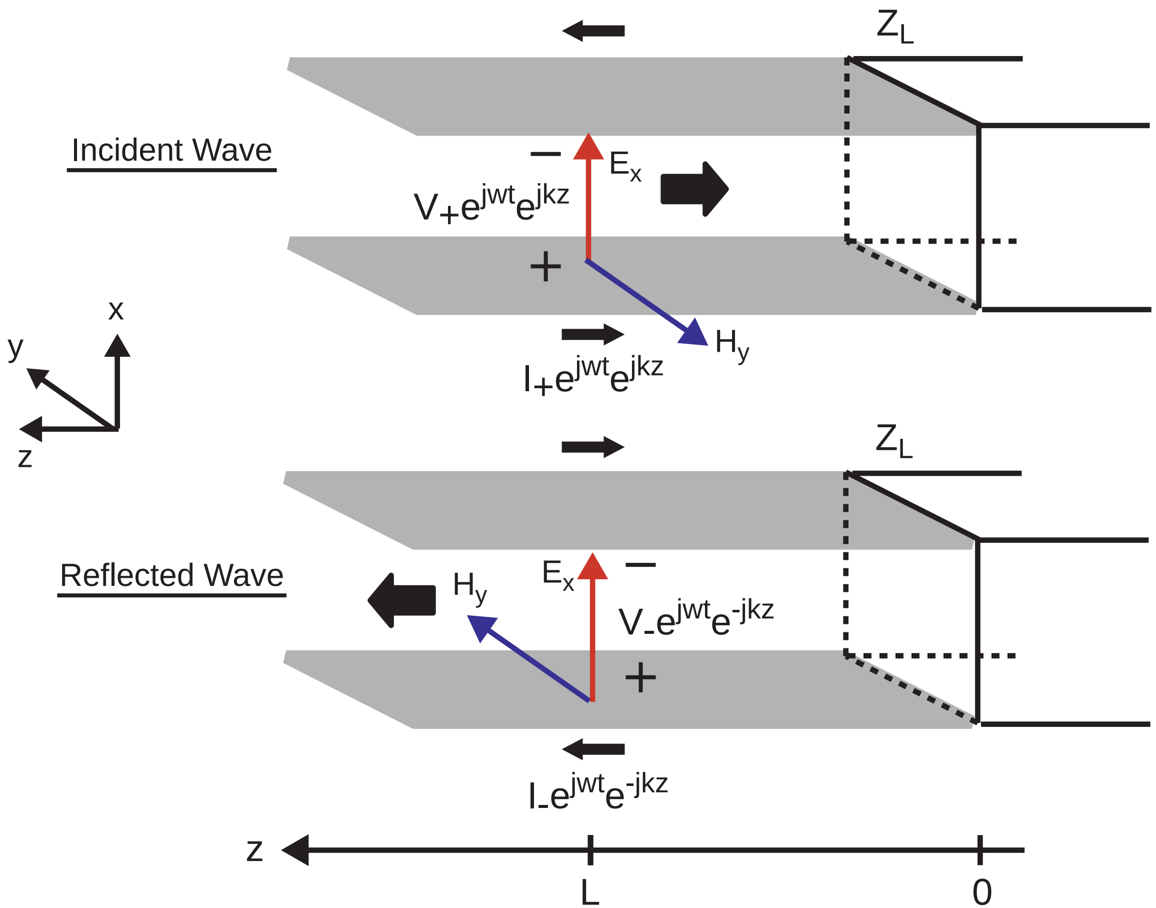
- Thus, the incident wave has the voltage $V_i(z,t) = V_+ e^{j(wt+kz)}$

- And the reflected wave $V_r(z,t) = V_- e^{j(wt-kz)}$

- Similarly for currents: $I_i(z,t) = I_+ e^{j(wt+kz)}$

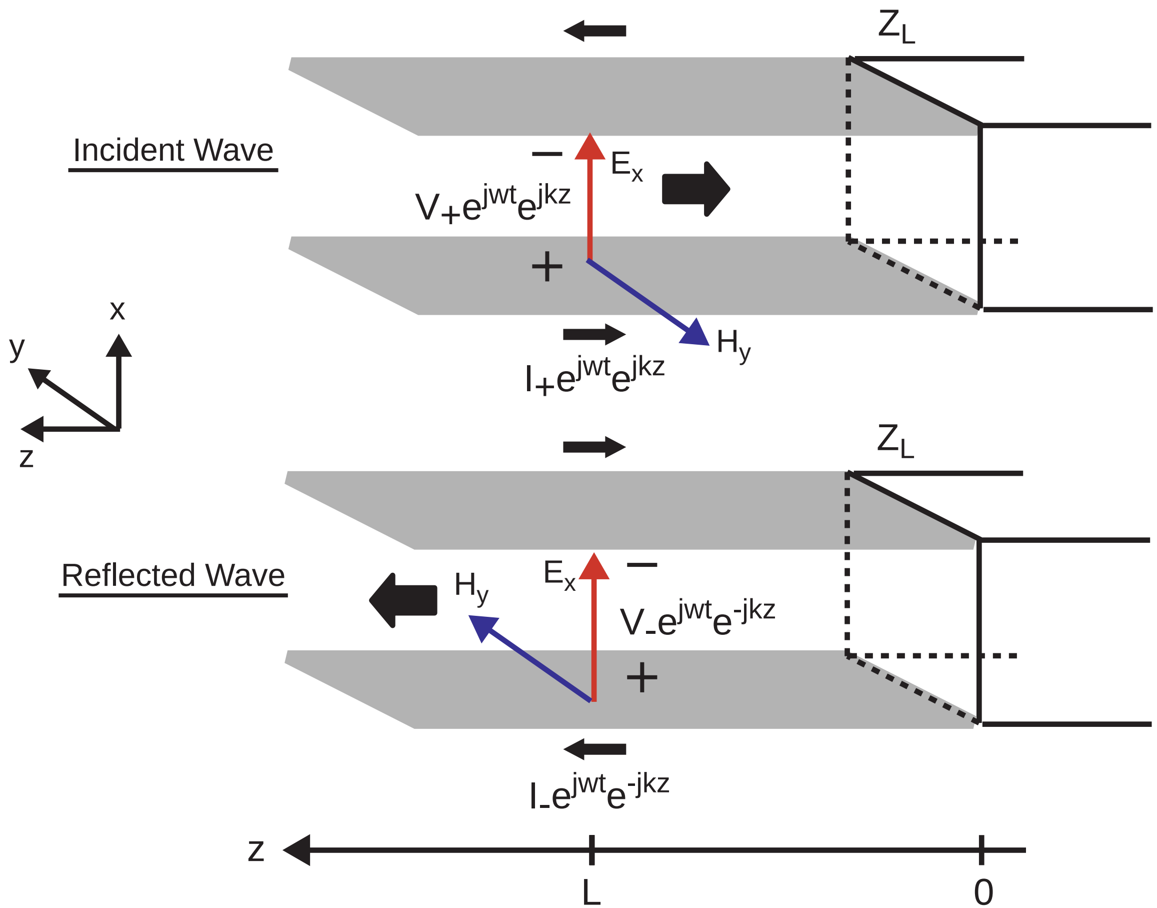
$$I_r(z,t) = I_- e^{j(wt-kz)}$$

Determine Voltage and Current At Different Positions



- **Incident and reflected waves must be added together**

Determine Voltage and Current At Different Positions



$$V(z,t) = V_+ e^{jwt} e^{jkz} + V_- e^{jwt} e^{-jkz}$$

$$I(z,t) = I_+ e^{jwt} e^{jkz} - I_- e^{jwt} e^{-jkz}$$

Define Generalized Reflection Coefficient

Recall: $\Gamma_L = \frac{V_r}{V_i}$

Define Generalized Reflection Coefficient: $\Gamma(z) = \frac{V_r(z,t)}{V_i(z,t)}$

$$\Gamma(z) = \frac{V_r(z,t)}{V_i(z,t)} = \frac{V_- e^{j\omega t} e^{-jkz}}{V + -e^{j\omega t} e^{jkz}} = \frac{V_-}{V_+} e^{-2jkz}$$

$$\Rightarrow \quad \Gamma(z) = \Gamma_L e^{-2jkz}$$

Since $\frac{V_i(z,t)}{I_i(z,t)} = \frac{V_r(z,t)}{I_r(z,t)} = Z_o$

$$\Gamma(z) = \frac{I_r(z,t)}{I_i(z,t)}$$

Generalized Reflection Coefficient Cont'd

$$V(z,t) = V_+ e^{jwt} e^{jkz} + V_- e^{jwt} e^{-jkz}$$

$$I(z,t) = I_+ e^{jwt} e^{jkz} - I_- e^{jwt} e^{-jkz}$$

$$V(z,t) = V_+ e^{jwt} e^{jkz} \left(1 + \frac{V_-}{V_+} e^{-2jkz}\right)$$

⬇

$$V(z,t) = V_+ e^{jwt} e^{jkz} \left(1 + \Gamma_L e^{-2jkz}\right)$$

⬇

$$V(z,t) = V_+ e^{jwt} e^{jkz} (1 + \Gamma(z))$$

Similarly:

$$I(z,t) = I_+ e^{jwt} e^{jkz} (1 - \Gamma(z))$$

A Closer Look at Γ(z)

$$\Gamma(z) = \Gamma_L e^{-2jkz}$$

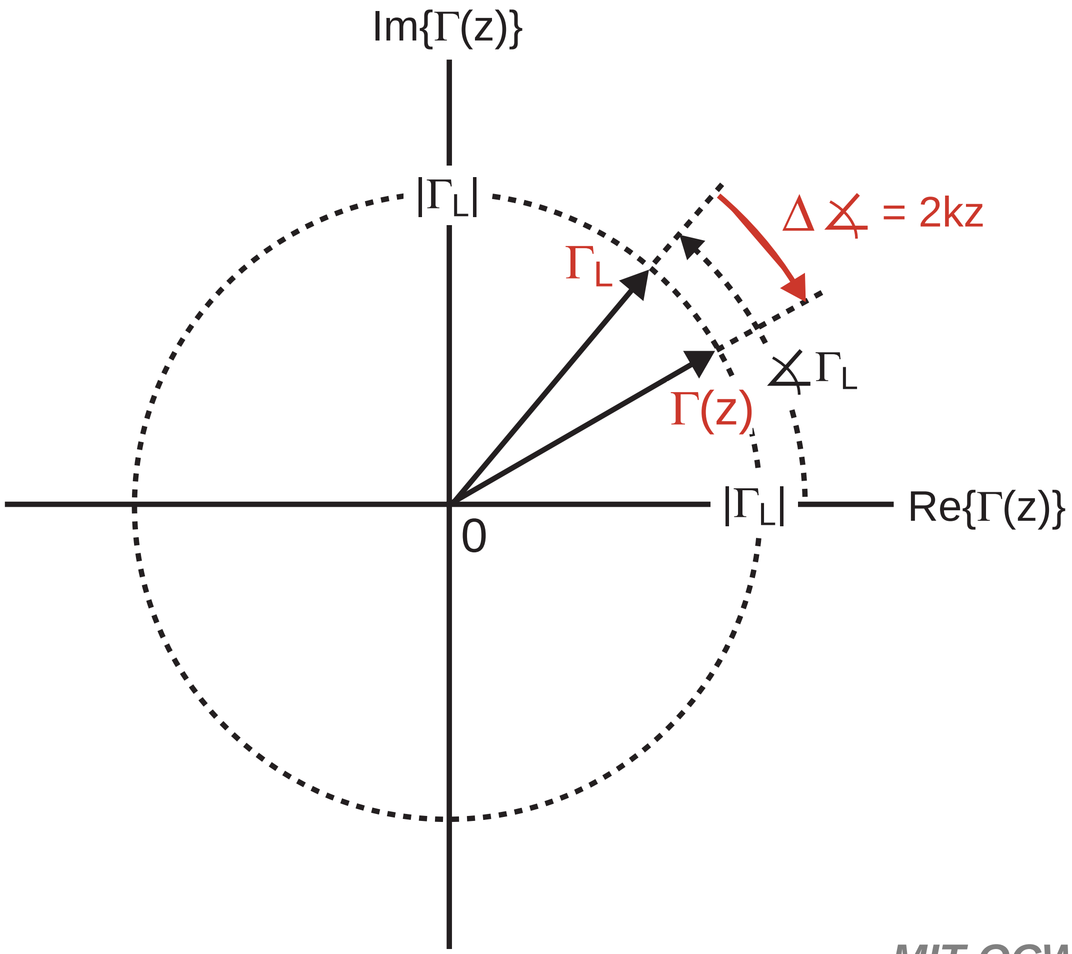
- **Recall Γ_L is**

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$$

Note: $|\Gamma_L| \leq 1$

for $Re\{Z_L/Z_o\} \geq 0$

- **We can view Γ(z) as a complex number that rotates clockwise as z (distance from the load) increases**



Calculate $|V_{max}|$ and $|V_{min}|$ Across The Transmission Line

- **We found that**

$$V(z,t) = V_+ e^{jwt} e^{jkz} (1 + \Gamma(z))$$

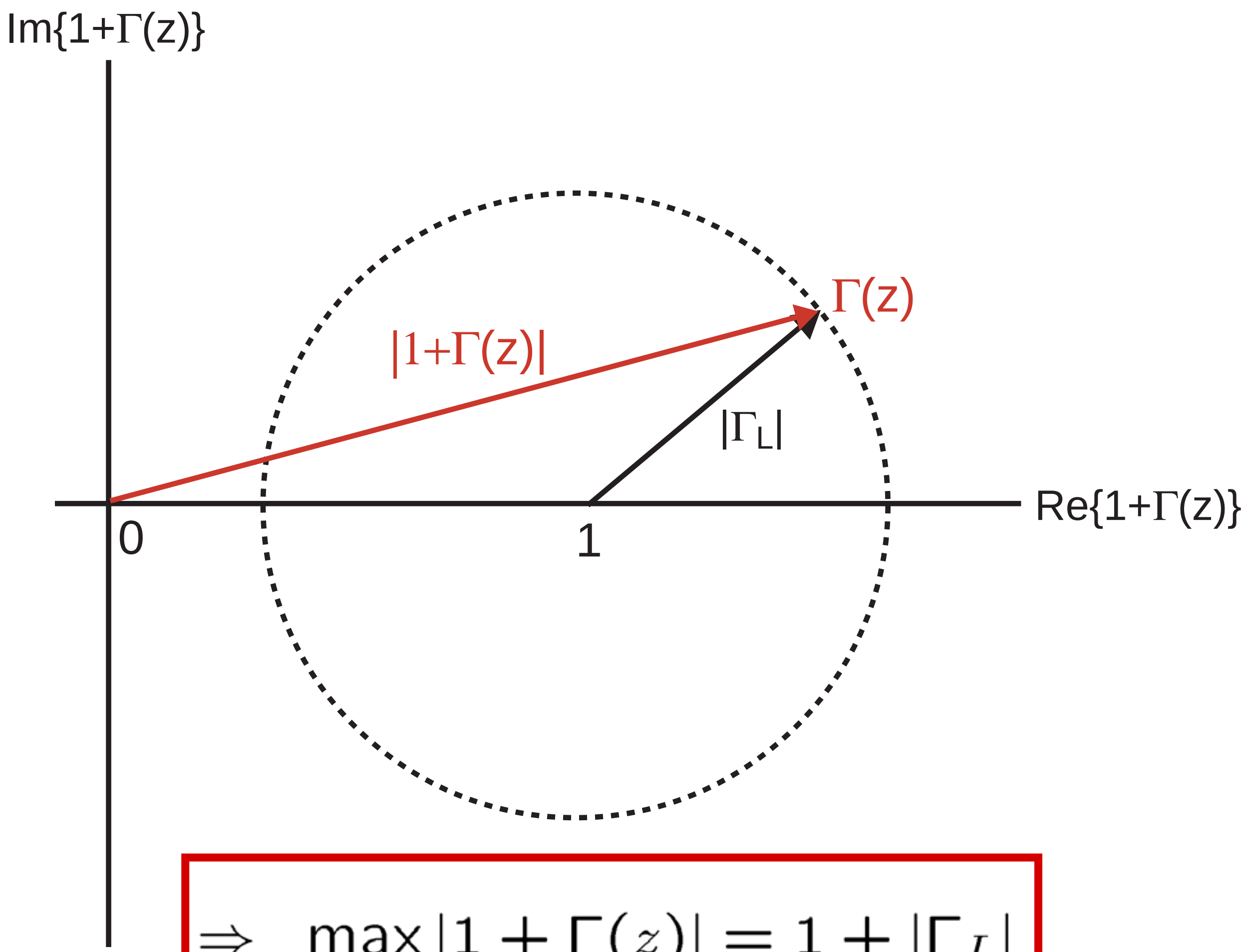
- **So that the max and min of V(z,t) are calculated as**

$$\Rightarrow \quad V_{max} = \max |V(z,t)| = |V_+| \max |1 + \Gamma(z)|$$

$$\Rightarrow \quad V_{min} = \min |V(z,t)| = |V_+| \min |1 + \Gamma(z)|$$

- **We can calculate this geometrically!**

A Geometric View of $|1+\Gamma(z)|$

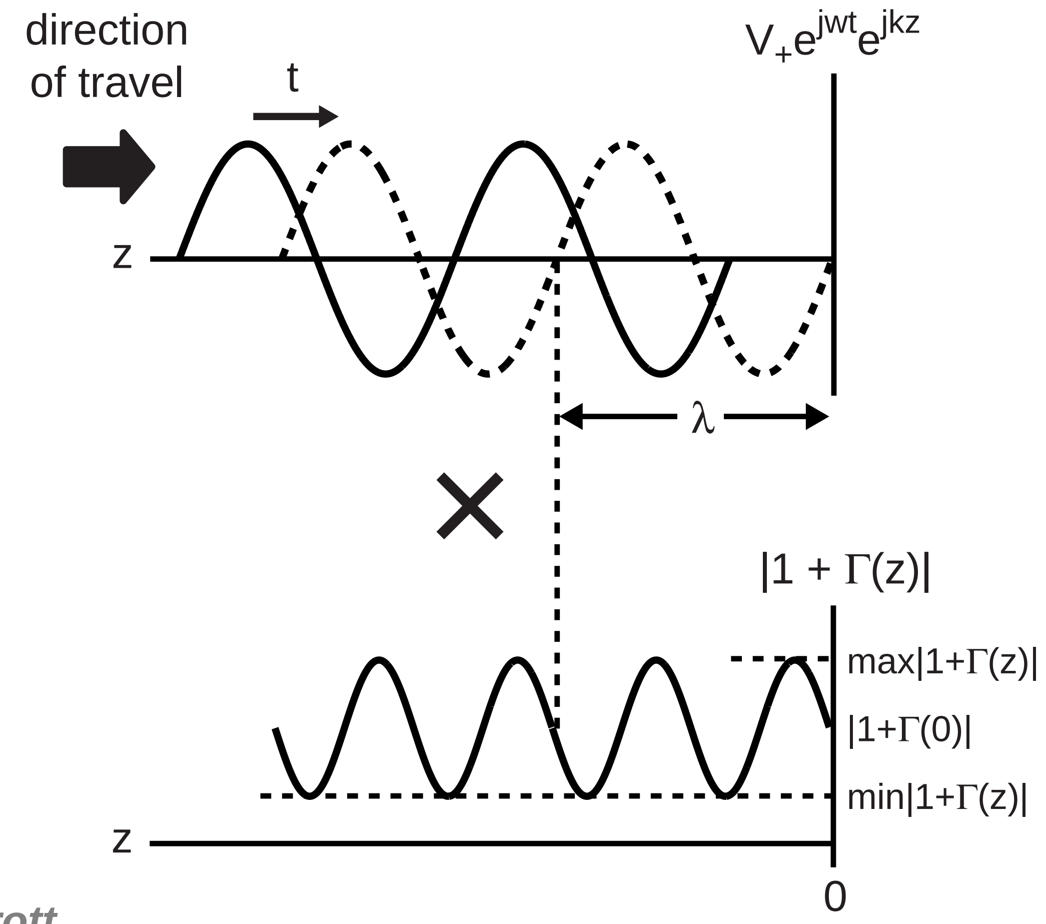


$$\Rightarrow \quad \max|1+\Gamma(z)| = 1+|\Gamma_L|$$

$$\Rightarrow \quad \min|1+\Gamma(z)| = 1-|\Gamma_L|$$

Reflections Cause Amplitude to Vary Across Line

- **Equation:** $V(z,t) = V_+ e^{jwt} e^{jkz} \left|1 + \Gamma(z)\right| e^{j\angle(1+\Gamma(z))}$
- **Graphical representation:**



Voltage Standing Wave Ratio (VSWR)

- **Definition**

$$\text{VSWR} = \frac{V_{max}}{V_{min}} = \frac{|V_+|(1+|\Gamma_L|)}{|V_+|(1-|\Gamma_L|)} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

- **For passive load (and line)**

$$|\Gamma_L| \leq 1 \quad \Rightarrow \quad 1 \leq \text{ VSWR } \leq \infty$$

$$|\Gamma_L| = 0 \qquad\qquad |\Gamma_L| = 1$$

- **We can infer the magnitude of the reflection coefficient based on VSWR**

$$|\Gamma_L| = \frac{\text{VSWR}-1}{\text{VSWR}+1}$$

Reflections Influence Impedance Across The Line

- **From Slide 7**

$$V(z,t) = V_+ e^{jwt} e^{jkz} (1 + \Gamma(z))$$

$$I(z,t) = I_+ e^{jwt} e^{jkz} (1 - \Gamma(z))$$

$$\Rightarrow \quad Z(z,t) = \frac{V_+(1+\Gamma(z))}{I_+(1-\Gamma(z))} = Z_o \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

  - **Note: not a function of time! (only of distance from load)**

- **Alternatively**

$$Z(z) = Z_o \frac{1+\Gamma_L e^{-2jkz}}{1-\Gamma_L e^{-2jkz}}$$

  - **From Lecture 3:** $\lambda = \frac{T}{\sqrt{\mu\epsilon}} = \frac{wT}{w\sqrt{\mu\epsilon}} = \frac{2\pi f T}{k} = \frac{2\pi}{k}$

Impedance as a Function of Location

We can now express Z(z) as

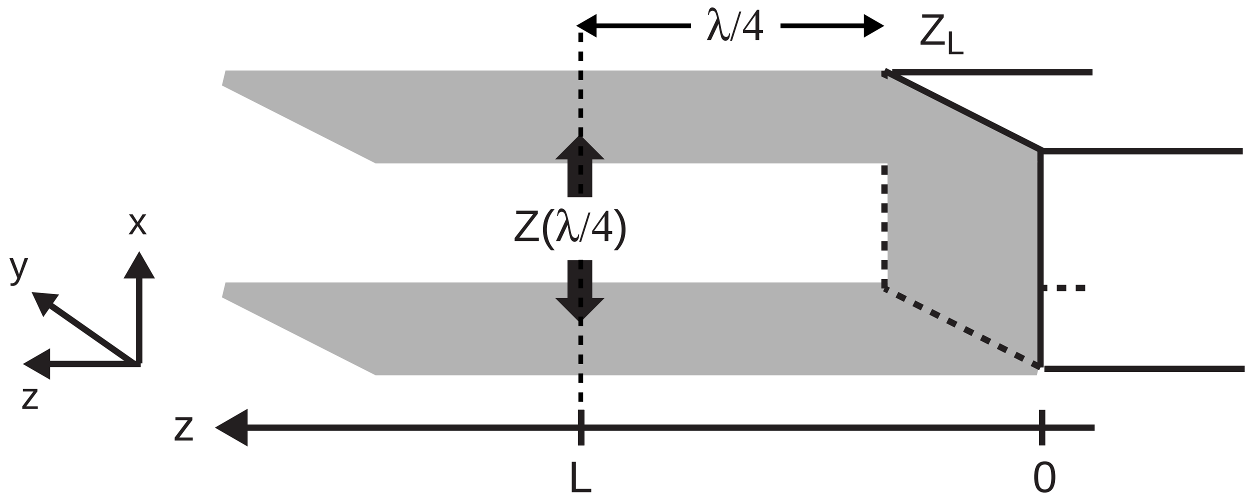
$$Z(z) = Z_o \frac{1 + \Gamma_L e^{-j(4\pi/\lambda)z}}{1 - \Gamma_L e^{-j(4\pi/\lambda)z}}$$

Also

$$\Gamma(z) = \Gamma_L e^{-j(4\pi/\lambda)z}$$

Note: Z(z) and Γ(z) are periodic in z with a period of λ/2

Example: Z(λ/4) with Shorted Load



- **Calculate reflection coefficient**

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0 - Z_o}{0 + Z_o} = -1$$

- **Calculate generalized reflection coefficient**

$$\Gamma(\lambda/4) = \Gamma_L e^{-j(4\pi/\lambda)(\lambda/4)} = \Gamma_L e^{-j\pi} = -\Gamma_L = 1$$

- **Calculate impedance**

$$Z(\lambda/4) = Z_o \frac{1 + \Gamma(z)}{1 - \Gamma(z)} = \infty \; !$$

Generalize Relationship Between Z(λ/4) and Z(0)

- **General formulation**

$$Z(z) = Z_o \frac{1 + \Gamma_L e^{-j(4\pi/\lambda)z}}{1 - \Gamma_L e^{-j(4\pi/\lambda)z}}$$

- **At load (z=0)**

$$Z_L = Z(0) = Z_o \frac{1 + \Gamma_L}{1 - \Gamma_L}$$

- **At quarter wavelength away (z = λ/4)**

$$Z(\lambda/4) = Z_o \frac{1 - \Gamma_L}{1 + \Gamma_L} = \frac{Z_o^2}{Z_L}$$

  - **Impedance is inverted (Z_o is real)**
    - Shorts turn into opens
    - Capacitors turn into inductors

Now Look At Z(Δ) (Impedance Close to Load)

- **Impedance formula (Δ very small)**

$$Z(\Delta) = Z_o \frac{1 + \Gamma_L e^{-2jk\Delta}}{1 - \Gamma_L e^{-2jk\Delta}}$$

  - **A useful approximation:** $e^{-jx} \approx 1 - jx$ for $x \ll 1$

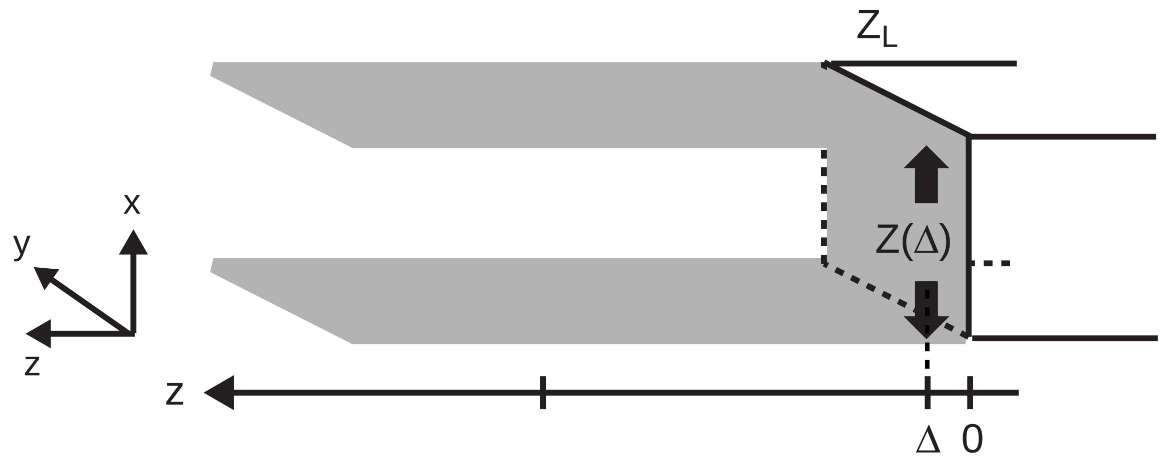
$$\Rightarrow \; e^{-2jk\Delta} \approx 1 - 2jk\Delta$$

  - **Recall from Lecture 2:** $k = w\sqrt{LC}, \quad Z_o = \sqrt{\frac{L}{C}}$

- **Overall approximation:**

$$Z(\Delta) \approx \left(\sqrt{\frac{L}{C}}\right) \frac{1 + \Gamma_L(1 - 2jw\sqrt{LC}\Delta)}{1 - \Gamma_L(1 - 2jw\sqrt{LC}\Delta)}$$

Example: Look At Z(Δ) With Load Shorted

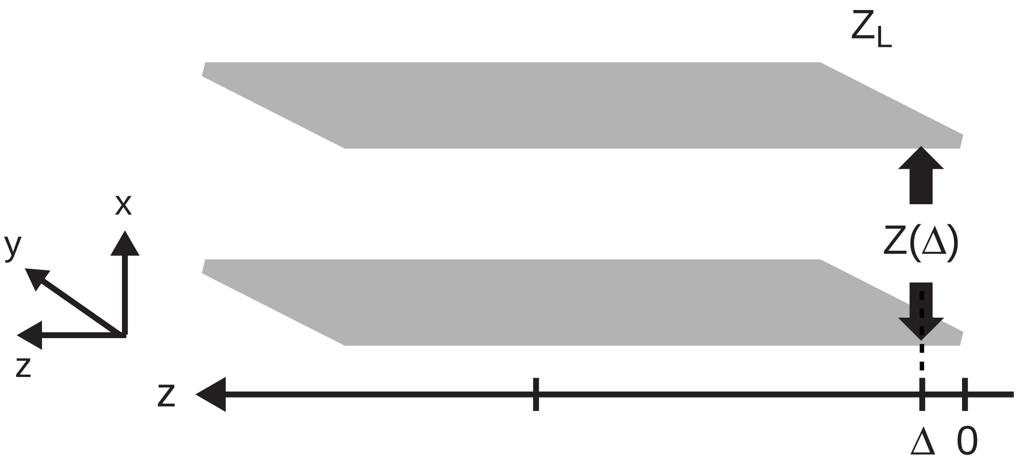


$$Z(\Delta) \approx \left(\sqrt{\frac{L}{C}}\right)\frac{1+\Gamma_L(1-2jw\sqrt{LC}\Delta)}{1-\Gamma_L(1-2jw\sqrt{LC}\Delta)}$$

- **Reflection coefficient:** $$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0 - Z_o}{0 + Z_o} = -1$$
- **Resulting impedance looks inductive!**

$$Z(\Delta) \approx \left(\sqrt{\frac{L}{C}}\right)\frac{1-(1-2jw\sqrt{LC}\Delta)}{1+(1-2jw\sqrt{LC}\Delta)} \approx jwL\Delta$$

Example: Look At $Z(\Delta)$ With Load Open

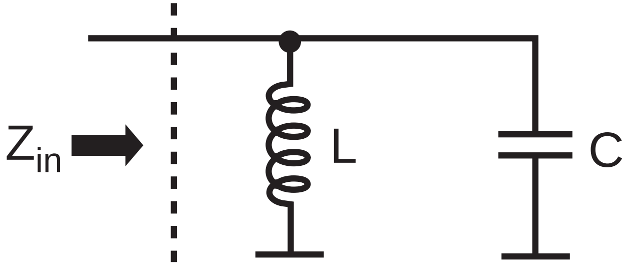


$$Z(\Delta) \approx \left(\sqrt{\frac{L}{C}}\right) \frac{1+\Gamma_L(1-2jw\sqrt{LC}\Delta)}{1-\Gamma_L(1-2jw\sqrt{LC}\Delta)}$$

- **Reflection coefficient:** $$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{\infty - Z_o}{\infty + Z_o} = 1$$
- **Resulting impedance looks capacitive!**

$$Z(\Delta) \approx \left(\sqrt{\frac{L}{C}}\right) \frac{1+(1-2jw\sqrt{LC}\Delta)}{1-(1-2jw\sqrt{LC}\Delta)} \approx \frac{1}{jwC\Delta}$$

Compare to an Ideal LC Tank Circuit



$$Z_{in}(w) = \frac{1}{jwC} || jwL = \frac{jwL}{1 - w^2LC}$$

- **Calculate input impedance about resonance**

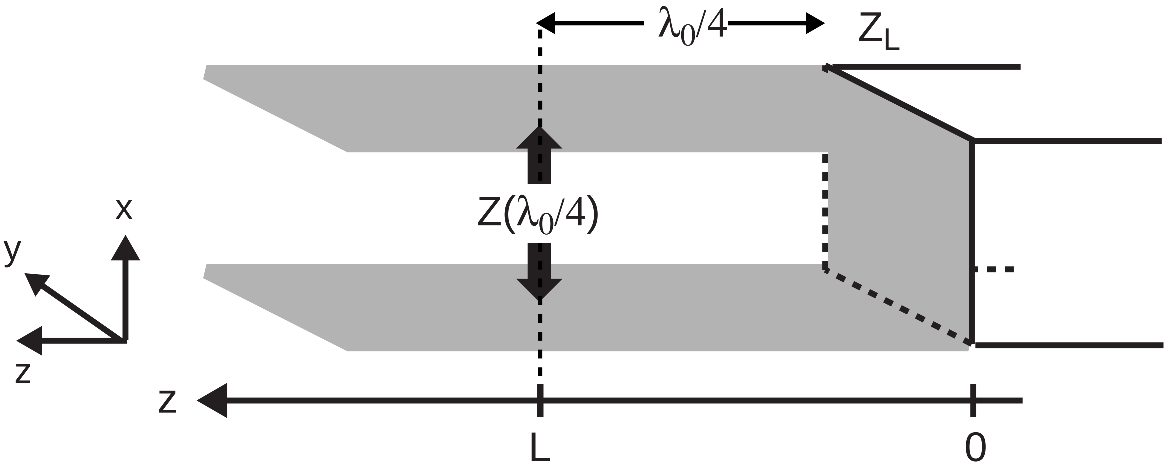
$$\text{Consider } w = w_o + \Delta w, \text{ where } w_o = \frac{1}{\sqrt{LC}}$$

$$Z_{in}(\Delta w) = \frac{j(w_o + \Delta w)L}{1 - (w_o + \Delta w)^2 LC}$$

$$= \frac{j(w_o + \Delta w)L}{\underbrace{1 - w_o^2 LC}_{=0} - 2\Delta w(w_oLC) - \underbrace{\Delta w^2 LC}_{\text{negligible}}}$$

$$\Rightarrow \; Z_{in}(\Delta w) \approx \frac{j(w_o + \Delta w)L}{-2\Delta w(w_oLC)} \approx \frac{jw_oL}{-2\Delta w(w_oLC)} = \boxed{-\frac{j}{2}\sqrt{\frac{L}{C}}\left(\frac{w_o}{\Delta w}\right)}$$

Transmission Line Version: $Z(\lambda_0/4)$ with Shorted Load



- **As previously calculated**

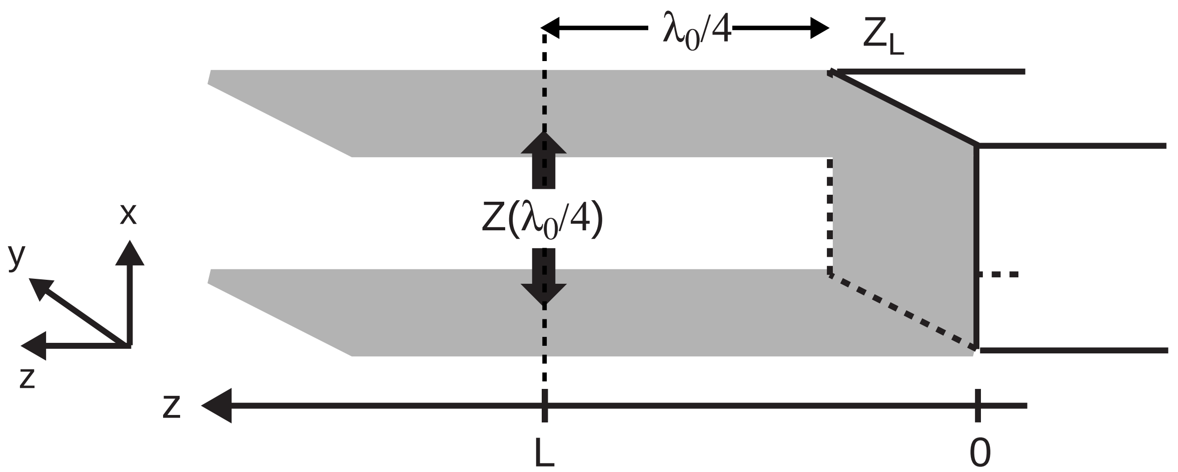
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0 - Z_o}{0 + Z_o} = -1$$

- **Impedance calculation**

$$Z(z) = Z_o \frac{1+\Gamma(z)}{1-\Gamma(z)}, \quad \text{where } \Gamma(z) = \Gamma_L e^{-j(4\pi/\lambda)z}$$

- **Relate λ to frequency** $\lambda = \dfrac{1}{f\sqrt{\mu\epsilon}} = \dfrac{1}{(f_o + \Delta f)\sqrt{\mu\epsilon}}$

Calculate Z(Δ f) – Step 1



- **Wavelength as a function of Δ f**

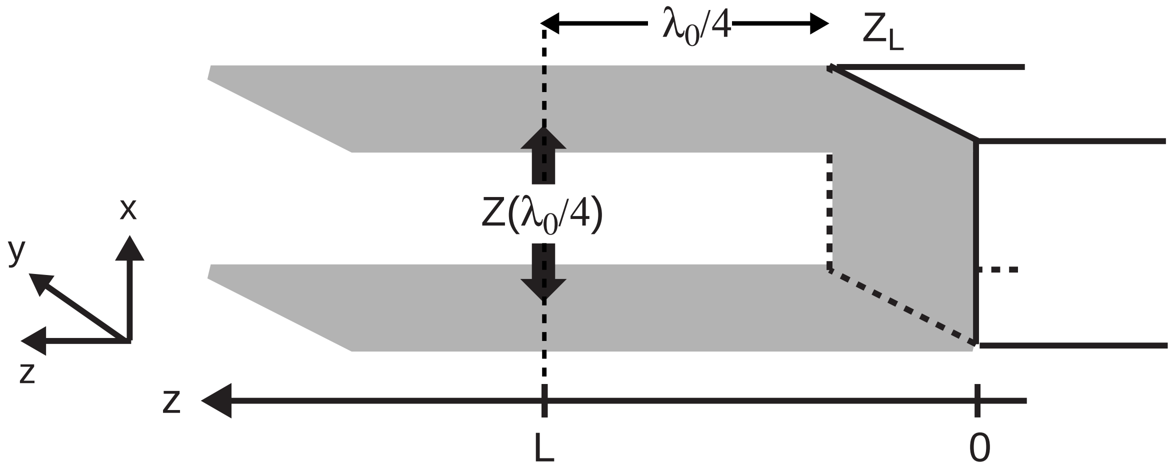
$$\lambda = \frac{1}{(f_o + \Delta f)\sqrt{\mu\epsilon}} = \frac{1}{f_o\sqrt{\mu\epsilon}(1 + \Delta f/f_o)} = \frac{\lambda_o}{1 + \Delta f/f_o}$$

- **Generalized reflection coefficient**

$$\Gamma(\lambda_o/4) = \Gamma_L e^{-j(4\pi/\lambda)\lambda_o/4} = \Gamma_L e^{-j\pi\lambda_o/\lambda} = \Gamma_L e^{-j\pi\lambda_o/\lambda}$$

$$\Rightarrow \ \Gamma(\lambda_o/4) = \Gamma_L e^{-j\pi(1+\Delta f/f_o)} = -\Gamma_L e^{-j\pi\Delta f/f_o}$$

Calculate Z(Δ f) – Step 2



- **Impedance calculation**

$$Z(\lambda_o/4) = Z_o \frac{1 - \Gamma_L e^{-j\pi\Delta f/f_o}}{1 + \Gamma_L e^{-j\pi\Delta f/f_o}} = Z_o \frac{1 + e^{-j\pi\Delta f/f_o}}{1 - e^{-j\pi\Delta f/f_o}}$$

- **Recall** $e^{-j\pi\Delta f/f_o} \approx 1 - j\pi\Delta f/f_o$

$$\Rightarrow \quad Z(z) \approx Z_o \frac{1 + 1 - j\pi\Delta f/f_o}{1 - 1 + j\pi\Delta f/f_o} \approx Z_o \frac{2}{j\pi\Delta f/f_o} = \boxed{-j\frac{2}{\pi}\sqrt{\frac{L}{C}}\left(\frac{w_o}{\Delta w}\right)}$$

  - **Looks like LC tank circuit (but more than one mode)!**

Smith Chart

- **Define normalized load impedance**

$$Z_n = \frac{Z_L}{Z_o}$$

- **Relation between Z_n and Γ_L**

$$Z_n = \frac{1+\Gamma_L}{1-\Gamma_L}$$

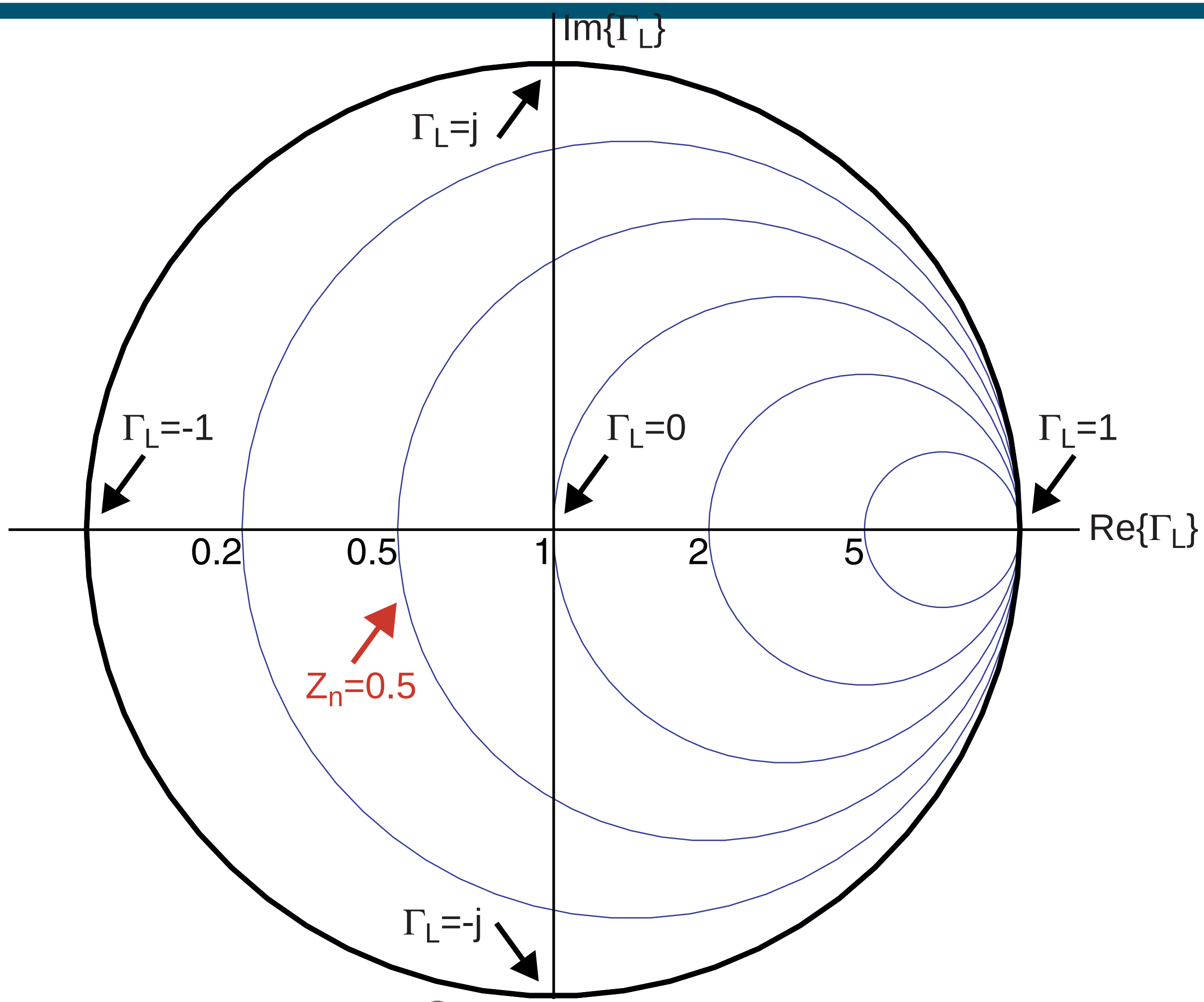
  - **Consider working in coordinate system based on Γ**

- **Key relationship between Z_n and Γ**

$$Re\{Z_n\} + jIm\{Z_n\} = \frac{1 + Re\{\Gamma_L\} + jIm\{\Gamma_L\}}{1 - (Re\{\Gamma_L\} + jIm\{\Gamma_L\})}$$

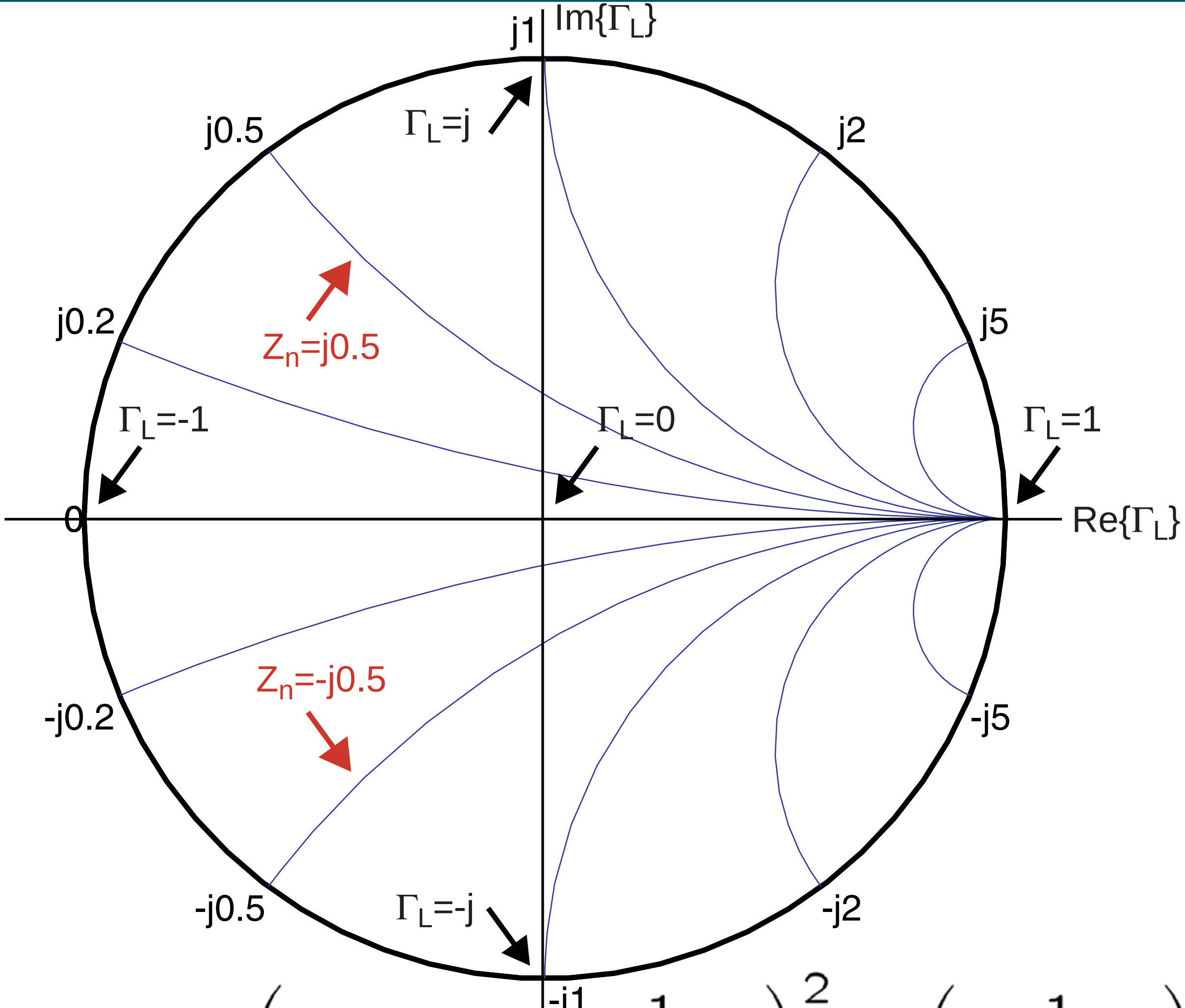
  - **Equate real and imaginary parts to get Smith Chart**

Real Impedance in Γ Coordinates (Equate Real Parts)



$$\left(Re\{\Gamma_L\} - \frac{Re\{Z_n\}}{1+Re\{Z_n\}}\right)^2 + (Im\{\Gamma_L\})^2 = \left(\frac{1}{1+Re\{Z_n\}}\right)^2$$

Imag. Impedance in Γ Coordinates (Equate Imag. Parts)



$$(Re\{\Gamma_L\} - 1)^2 + \left(Im\{\Gamma_L\} - \frac{1}{Im\{Z_n\}}\right)^2 = \left(\frac{1}{Im\{Z_n\}}\right)^2$$

What Happens When We Invert the Impedance?

- **Fundamental formulas**

$$Z_n = \frac{1+\Gamma_L}{1-\Gamma_L} \quad \Rightarrow \quad \Gamma_L = \frac{Z_n - 1}{Z_n + 1}$$

- **Impact of inverting the impedance**

$$Z_n \rightarrow 1/Z_n \quad \Rightarrow \quad \Gamma_L \rightarrow -\Gamma_L$$

  - **Derivation:**

$$\frac{1/Z_n - 1}{1/Z_n + 1} = \frac{1 - Z_n}{1 + Z_n} = -\left(\frac{Z_n - 1}{Z_n + 1}\right)$$

- **We can invert complex impedances in Γ plane by simply changing the sign of Γ !**

- **How can we best exploit this?**

The Smith Chart as a Calculator for Matching Networks

- **Consider constructing both impedance and admittance curves on Smith chart**

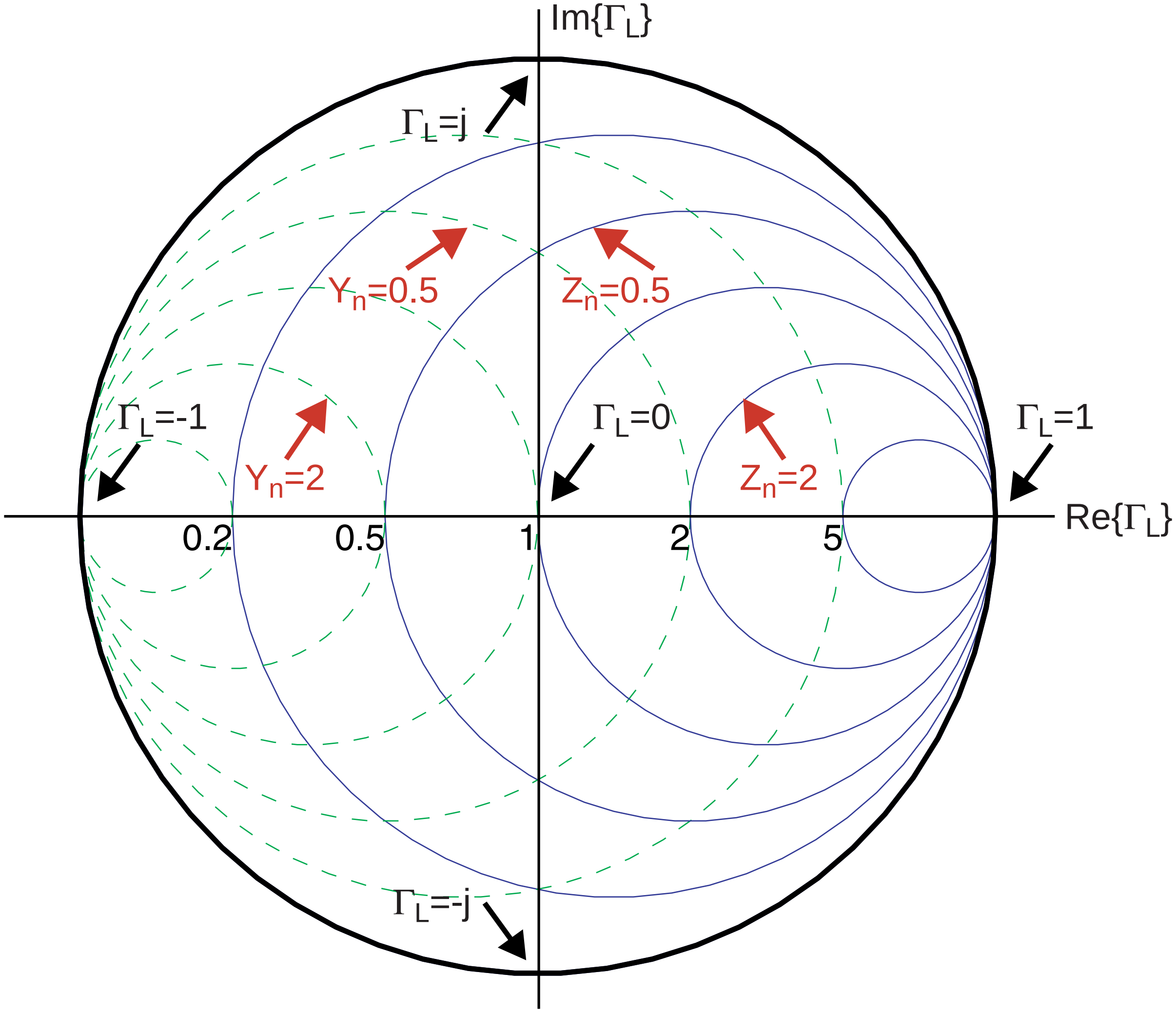
$$Z_n \rightarrow 1/Z_n \;\; \Rightarrow \;\; \Gamma_L \rightarrow -\Gamma_L$$

  - **Conductance curves derived from resistance curves**
  - **Susceptance curves derived from reactance curves**

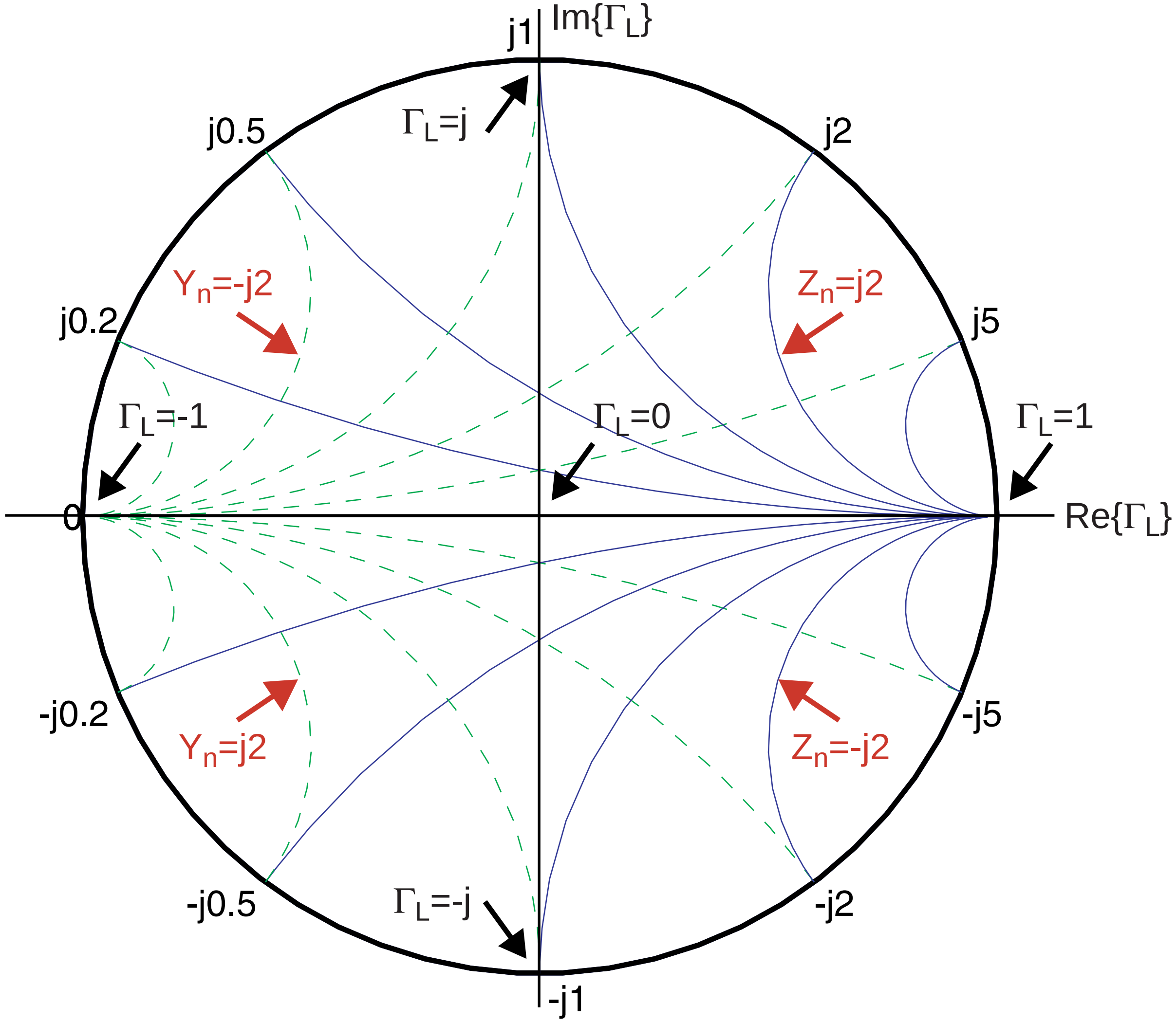
- **For series circuits, work with impedance**
  - **Impedances add for series circuits**

- **For parallel circuits, work with admittance**
  - **Admittances add for parallel circuits**

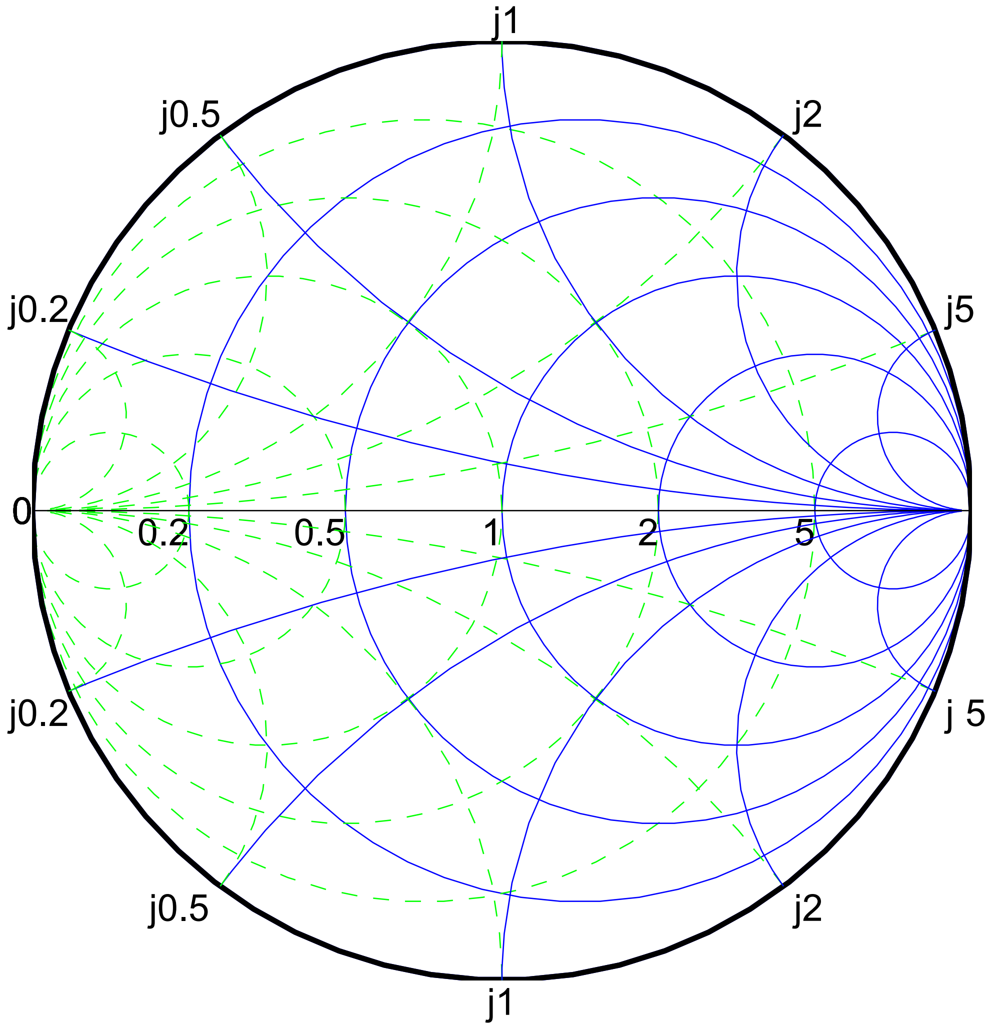
Resistance and Conductance on the Smith Chart



Reactance and Susceptance on the Smith Chart



Overall Smith Chart



Smith Chart element Paths

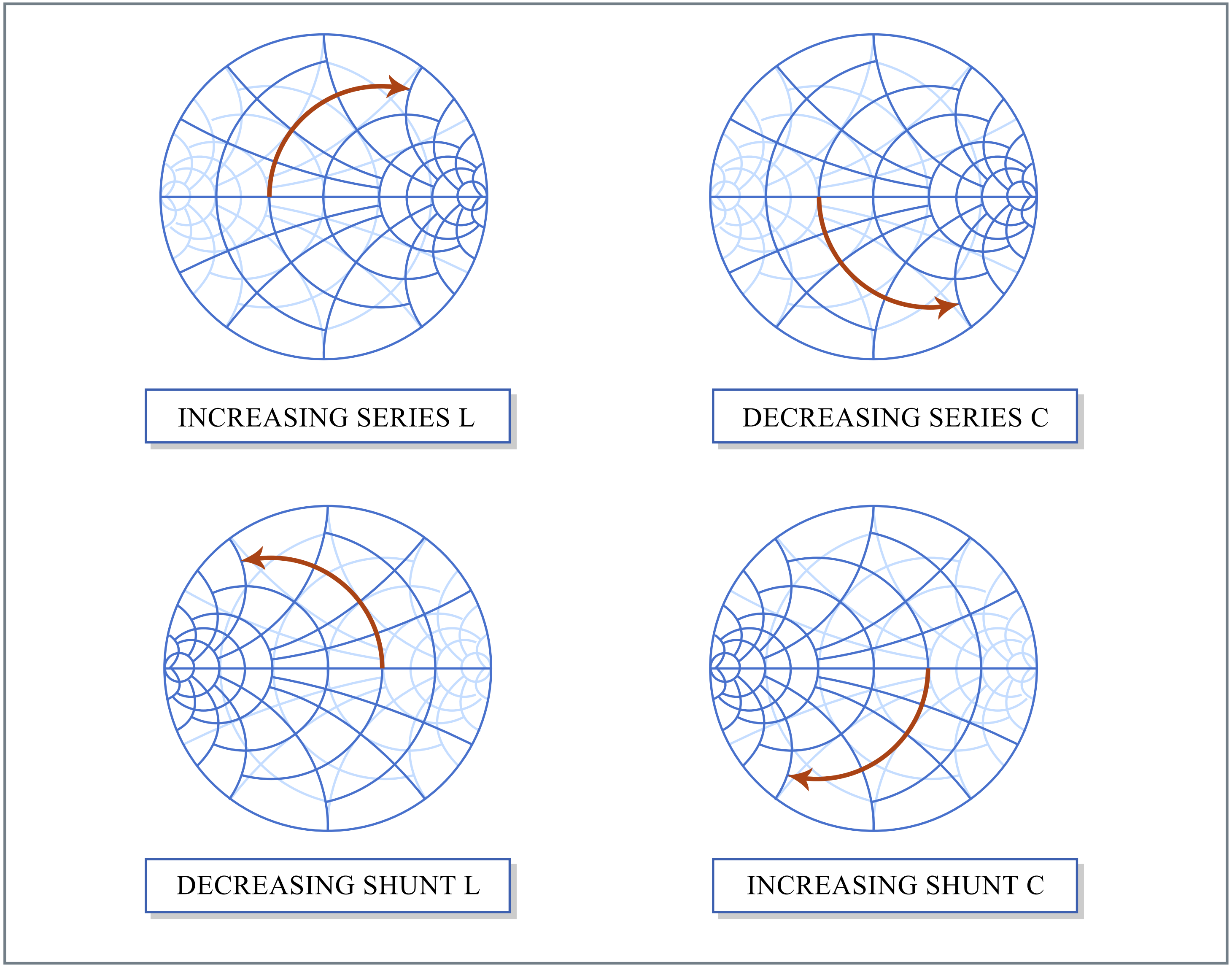
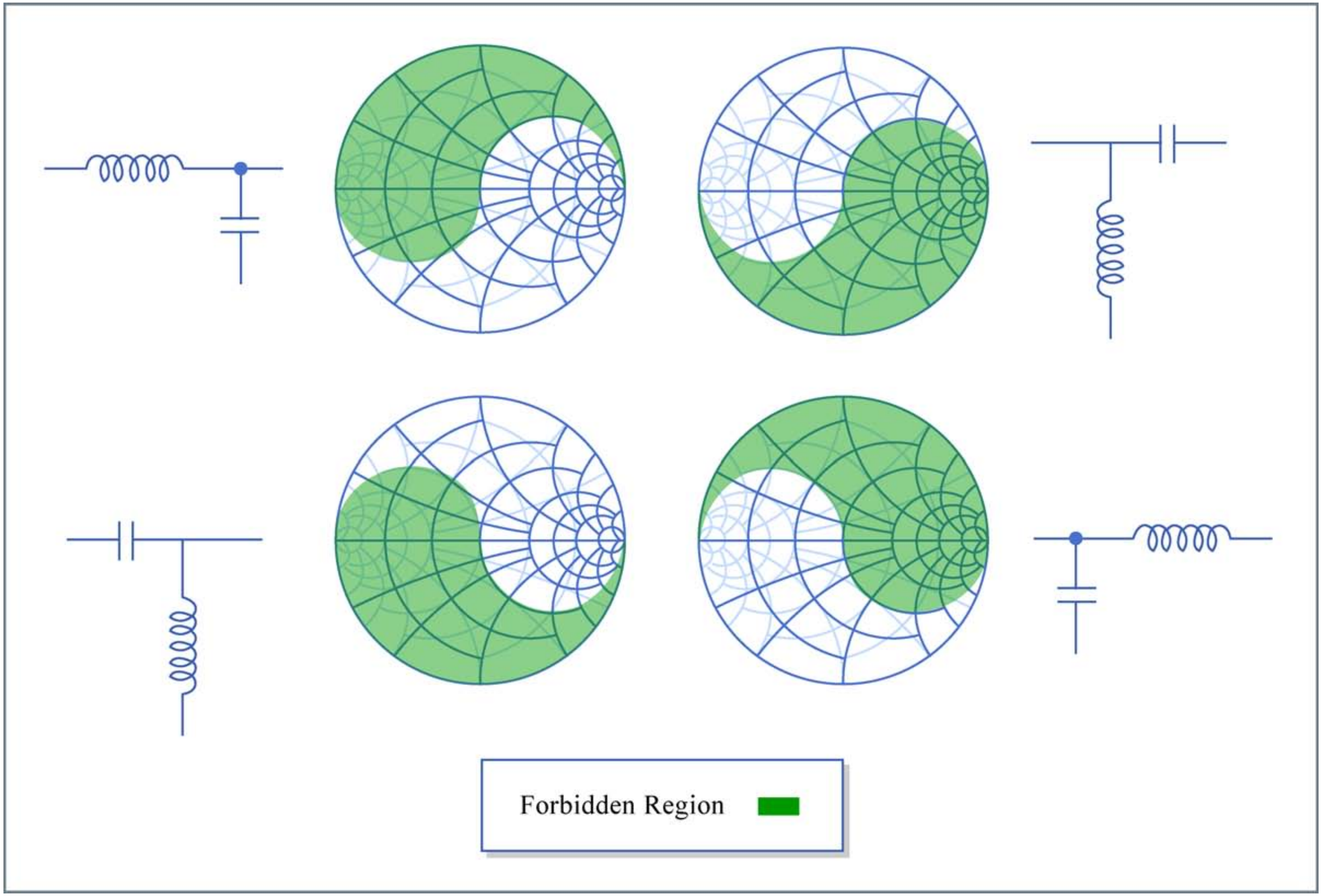


Figure by MIT OCW. Adapted from
http://contact.tm.agilent.com/Agilent/tmo/an-95-1/classes/imatch.html

L-Match Circuits (Matching Load to Source)

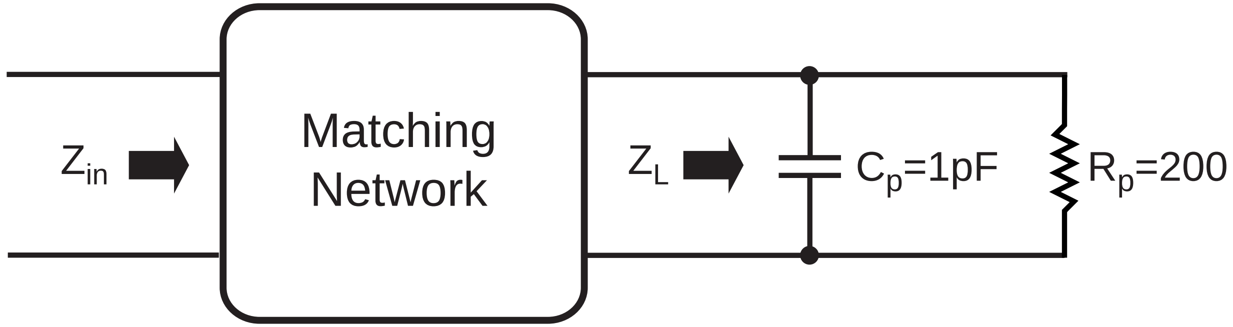


π or L match networks
remove forbidden regions

Figure by MIT OCW. Adapted from
http://contact.tm.agilent.com/Agilent/tmo/an-95-1/classes/imatch.html

Example – Match RC Network to 50 Ohms at 2.5 GHz

- **Circuit**



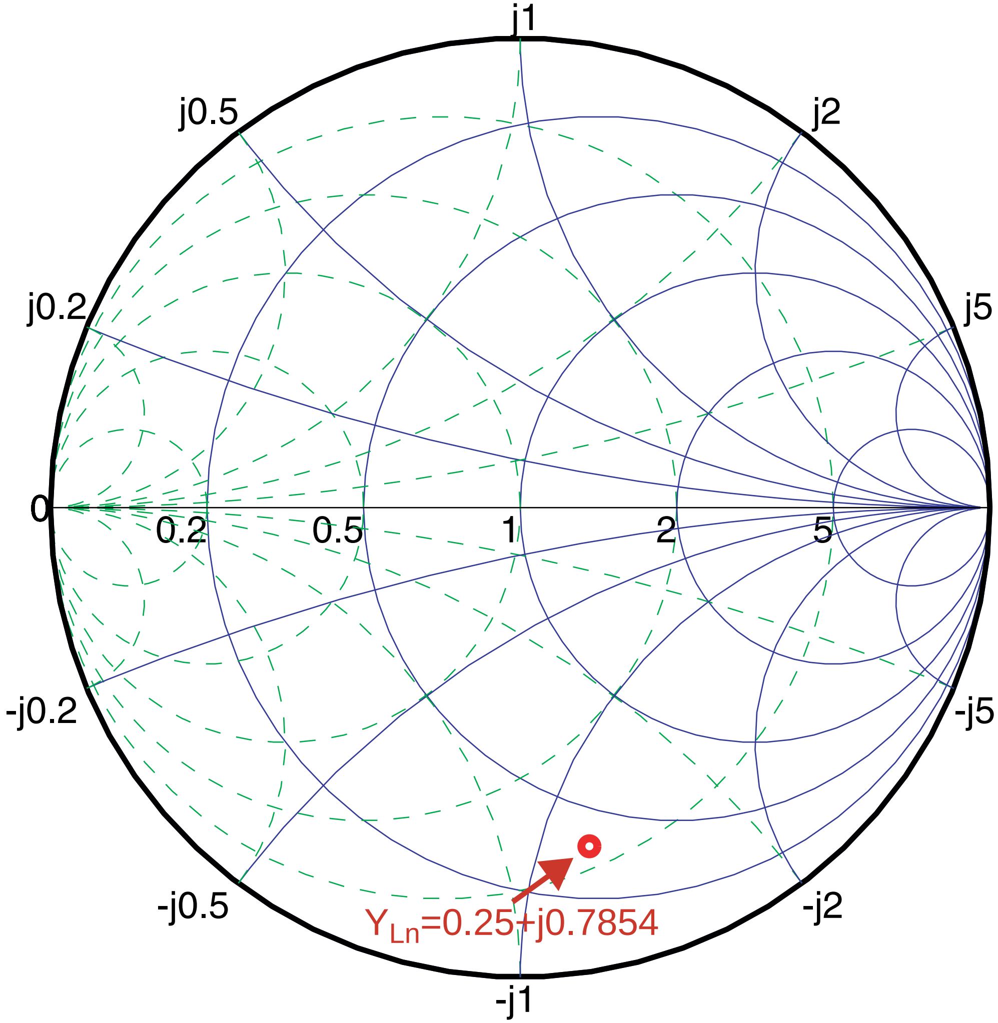
- **Step 1: Calculate Z_{Ln}**

$$Z_{Ln} = \frac{Z_L}{Z_o} = \frac{R_L||(1/jwC)}{50} = \frac{1}{50(1/R_L + jwC)}$$

$$= \frac{1}{50(1/200 + j2\pi(2.5e9)10^{-12})} = \frac{1}{0.25 + j.7854}$$

- **Step 2: Plot Z_{Ln} on Smith Chart (use admittance, Y_{Ln})**

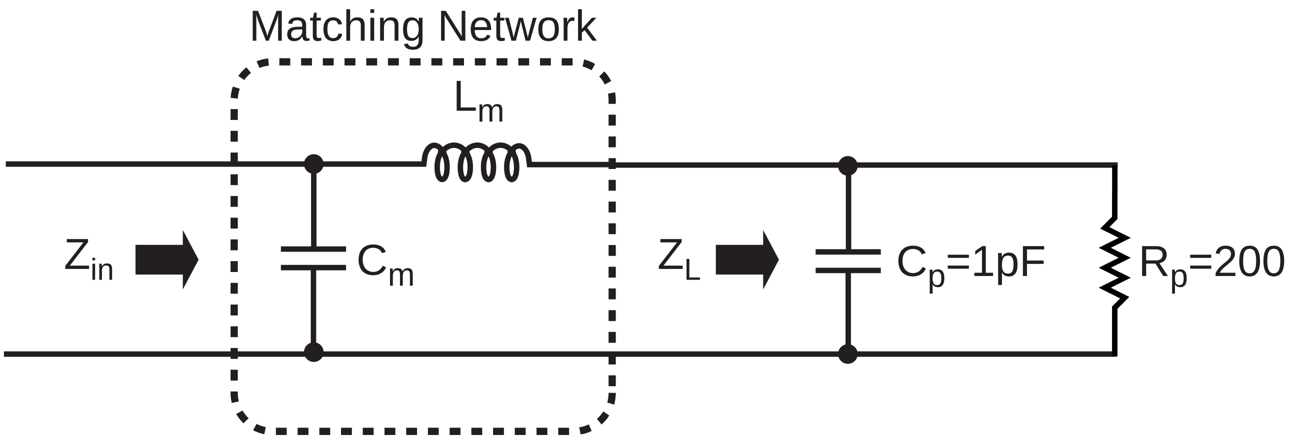
Plot Starting Impedance (Admittance) on Smith Chart



(Note: Z_{Ln}=0.37-j1.16)

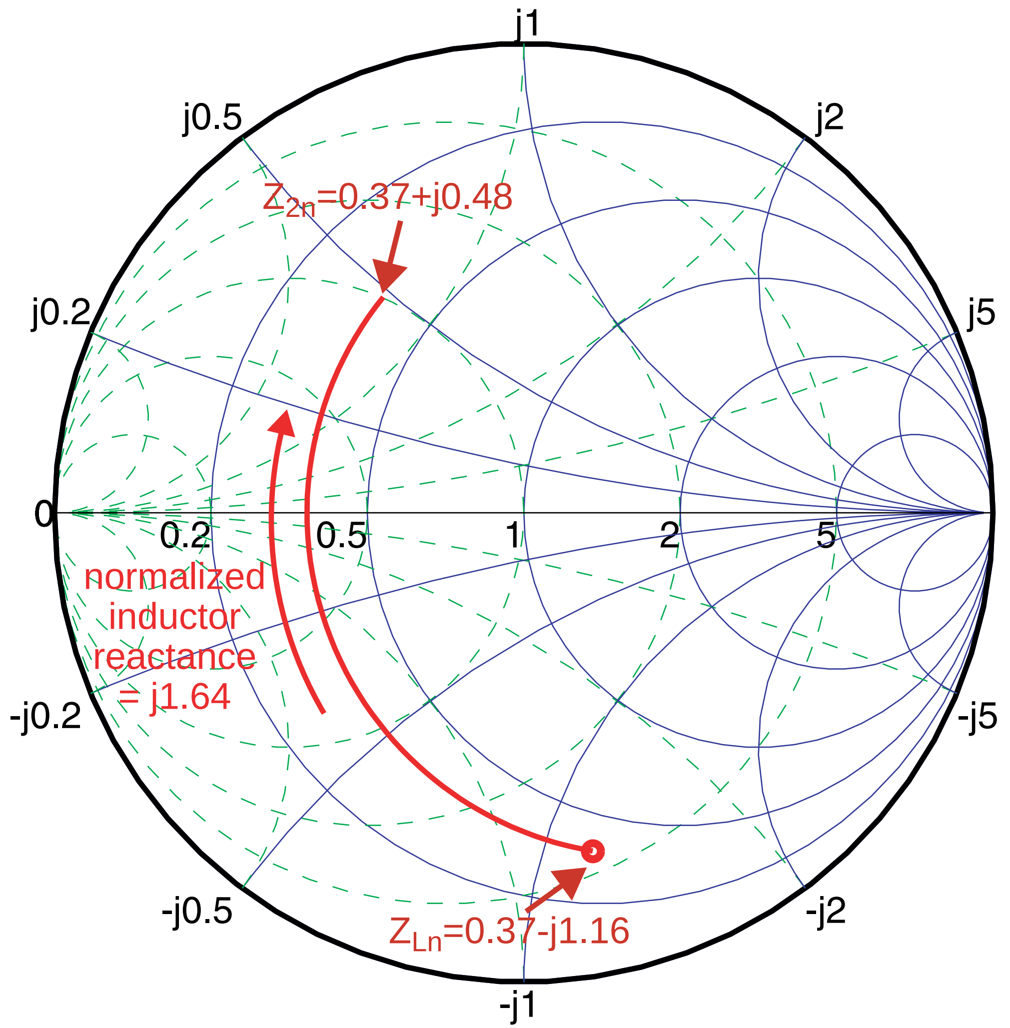
Develop Matching "Game Plan" Based on Smith Chart

- By inspection, we see that the following matching network can bring us to Z_{in} = 50 Ohms (center of Smith chart)-need to *step up* impedance



- Use the Smith chart to come up with component values
  - Inductance L_m shifts impedance up along reactance curve
  - Capacitance C_m shifts impedance down along susceptance curve

Add Reactance of Inductor L_m



Inductor Value Calculation Using Smith Chart

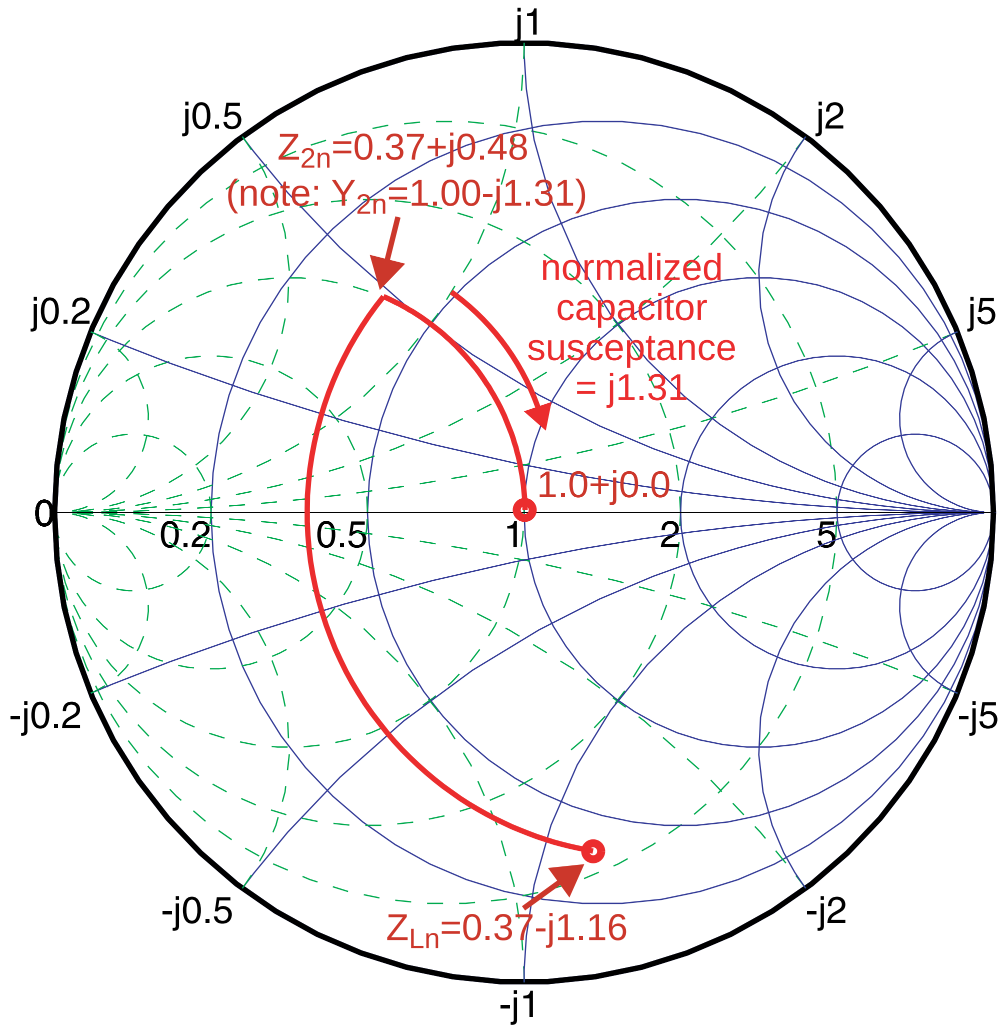
- **From Smith chart, we found that the desired normalized inductor reactance is**

$$\frac{jwL_m}{Z_o} = \frac{jwL_m}{50} = j1.64$$

- **Required inductor value is therefore**

$$\Rightarrow \quad L_m = \frac{50(1.64)}{2\pi 2.5e9} = 5.2nH$$

Add Susceptance of Capacitor C_m (Achieves Match!)



Capacitor Value Calculation Using Smith Chart

- **From Smith chart, we found that the desired normalized capacitor susceptance is**

$$Z_o jwC_m = 50jwC_m = j1.31$$

- **Required capacitor value is therefore**

$$\Rightarrow \quad C_m = \frac{1.31}{50(2\pi 2.5e9)} = 1.67pF$$

Useful Web Resource

- **Play the "matching game" at**

http://contact.tm.agilent.com/Agilent/tmo/an-95-1/classes/imatch.html

  - **Allows you to graphically tune several matching networks**
  - **Note: it is set up to match *source* impedance to load impedance rather than match the load to the source impedance**
    - Same results, just different viewpoint