CHAPTER A

Robotics Preliminaries

A.1 DERIVING THE EQUATIONS OF MOTION (AN EXAMPLE)

The equations of motion for a standard robot can be derived using the method of Lagrange. Using T as the total kinetic energy of the system, and U as the total potential energy of the system, $L = T - U$, and Q_i as the generalized force corresponding to q_i, the Lagrangian dynamic equations are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i.$$

If you are not comfortable with these equations, then any good book chapter on rigid body mechanics can bring you up to speed[1]; for now you can take them as a handle that you can crank to generate equations of motion.

EXAMPLE A.1 Simple Double Pendulum

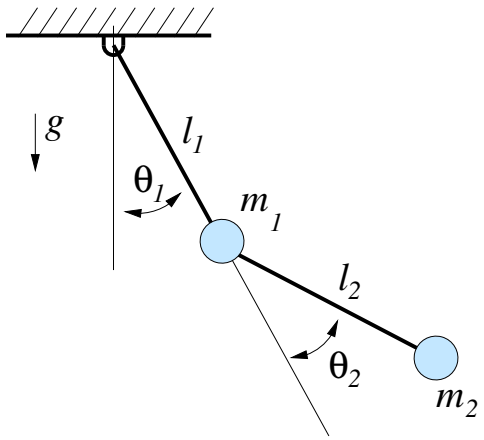

FIGURE A.1 Simple Double Pendulum

Consider the system in Figure A.1 with torque actuation at both joints, and all of the mass concentrated in two points (for simplicity). Using $\mathbf{q} = [\theta_1, \theta_2]^T$, and $\mathbf{x}_1, \mathbf{x}_2$ to

[1] Try [27] for a very practical guide to robot kinematics/dynamics, [35] for a hard-core dynamics text or [85] for a classical dynamics text which is a nice read

denote the locations of m_1, m_2, respectively, the kinematics of this system are:

$$\begin{aligned}\mathbf{x}_1 = &\begin{bmatrix} l_1 s_1 \\ -l_1 c_1 \end{bmatrix}, \quad \mathbf{x}_2 = \mathbf{x}_1 + \begin{bmatrix} l_2 s_{1+2} \\ -l_2 c_{1+2} \end{bmatrix} \\ \dot{\mathbf{x}}_1 = &\begin{bmatrix} l_1 \dot{q}_1 c_1 \\ l_1 \dot{q}_1 s_1 \end{bmatrix}, \quad \dot{\mathbf{x}}_2 = \dot{\mathbf{x}}_1 + \begin{bmatrix} l_2(\dot{q}_1 + \dot{q}_2) c_{1+2} \\ l_2(\dot{q}_1 + \dot{q}_2) s_{1+2} \end{bmatrix}\end{aligned}$$

Note that s_1 is shorthand for $sin(q_1)$, c_{1+2} is shorthand for $cos(q_1 + q_2)$, etc. From this we can easily write the kinetic and potential energy:

$$\begin{aligned}T =& \frac{1}{2}\dot{\mathbf{x}}_1^T m_1 \dot{\mathbf{x}}_1 + \frac{1}{2}\dot{\mathbf{x}}_2^T m_2 \dot{\mathbf{x}}_2 \\ =& \frac{1}{2}(m_1 + m_2) l_1^2 \dot{q}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{q}_1 + \dot{q}_2)^2 + m_2 l_1 l_2 \dot{q}_1 (\dot{q}_1 + \dot{q}_2) c_2 \\ U =& m_1 g y_1 + m_2 g y_2 = -(m_1 + m_2) g l_1 c_1 - m_2 g l_2 c_{1+2}\end{aligned}$$

Taking the partial derivatives $\frac{\partial T}{\partial q_i}$, $\frac{\partial T}{\partial \dot{q}_i}$, and $\frac{\partial U}{\partial q_i}$ ($\frac{\partial U}{\partial \dot{q}_i}$ terms are always zero), then $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}$, and plugging them into the Lagrangian, reveals the equations of motion:

$$\begin{aligned}&(m_1 + m_2) l_1^2 \ddot{q}_1 + m_2 l_2^2 (\ddot{q}_1 + \ddot{q}_2) + m_2 l_1 l_2 (2\ddot{q}_1 + \ddot{q}_2) c_2 \\ &\qquad - m_2 l_1 l_2 (2\dot{q}_1 + \dot{q}_2) \dot{q}_2 s_2 + (m_1 + m_2) l_1 g s_1 + m_2 g l_2 s_{1+2} = \tau_1 \\ &m_2 l_2^2 (\ddot{q}_1 + \ddot{q}_2) + m_2 l_1 l_2 \ddot{q}_1 c_2 + m_2 l_1 l_2 \dot{q}_1^2 s_2 + m_2 g l_2 s_{1+2} = \tau_2\end{aligned}$$

Numerically integrating (and animating) these equations in MATLAB produces the expected result.

A.2 THE MANIPULATOR EQUATIONS

If you crank through the Lagrangian dynamics for a few simple serial chain robotic manipulators, you will begin to see a pattern emerge - the resulting equations of motion all have a characteristic form. For example, the kinetic energy of your robot can always be written in the form:

$$T = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{H}(\mathbf{q}) \dot{\mathbf{q}},$$

where $\mathbf{H}$ is the state-dependent inertial matrix. This abstraction affords some insight into general manipulator dynamics - for example we know that $\mathbf{H}$ is always positive definite, and symmetric[7, p.107].

Continuing our abstractions, we find that the equations of motion of a general robotic manipulator take the form

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{B}(\mathbf{q})\mathbf{u},$$

where $\mathbf{q}$ is the state vector, $\mathbf{H}$ is the inertial matrix, $\mathbf{C}$ captures Coriolis forces, and $\mathbf{G}$ captures potentials (such as gravity). The matrix $\mathbf{B}$ maps control inputs $\mathbf{u}$ into generalized forces.

EXAMPLE A.2 Manipulator Equation form of the Simple Double Pendulum

The equations of motion from Example 1 can be written compactly as:

$$\mathbf{H}(\mathbf{q}) = \begin{bmatrix} (m_1+m_2)l_1^2 + m_2l_2^2 + 2m_2l_1l_2c_2 & m_2l_2^2 + m_2l_1l_2c_2 \\ m_2l_2^2 + m_2l_1l_2c_2 & m_2l_2^2 \end{bmatrix}$$

$$\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} 0 & -m_2l_1l_2(2\dot{q}_1+\dot{q}_2)s_2 \\ m_2l_1l_2\dot{q}_1s_2 & 0 \end{bmatrix}$$

$$\mathbf{G}(\mathbf{q}) = g\begin{bmatrix} (m_1+m_2)l_1s_1 + m_2l_2s_{1+2} \\ m_2l_2s_{1+2} \end{bmatrix}$$

Note that this choice of the $\mathbf{C}$ matrix was not unique.

The manipulator equations are very general, but they do define some important characteristics. For example, $\ddot{\mathbf{q}}$ is (state-dependent) linearly related to the control input, $\mathbf{u}$. This observation justifies the form of the dynamics assumed in equation 1.1.

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Spring 2009

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