**PROFESSOR:** I'm excited about today's lecture because there's so many fun topics. This is like many fun things all in one lecture. We're going to start with a cool little problem, which is about unfolding and re-folding. You could think of them-- they're kind of like hinged dissections although they're from between surfaces of polyhedra. You can also think of them as common unfoldings.

So general idea is, you have over here in the left team you have unfoldings-- polygons, let's say. Possible things that could fold into polyhedra. And we'll think convex again. And when we were thinking about unfolding, we we're thinking about going this direction. When we were thinking about folding, we were thinking about this direction. But what if we do both repeatedly?

So I start, say, with a cube. And then I unfold it and then I re-fold it. That's sort of what the metamorphosis of the cube was about. And then I unfold that and I re-fold it, and I unfold and I re-fold and I unfold and re-fold. This is a fun tour to take. Is this space connected? Can I get from any polyhedral surface to any other-- say, convex-- by unfolding and re-folding? We have no idea.

So while each of these operations we know a fair amount about-- we know how to generally unfold convex polyhedra, we know how to find all of the gluings of a polygon into a polyhedron-- repeating those processes, we could explore but actually no ones' every tried that algorithmically. It'd be fun to see what crazy things you get.

But we have lots of examples and partial progress towards this question. So here's the first example. This appears in the book. It's by Hirata from 2000. He was one of the guys who implemented a gluing algorithm. And this is an unfolding-- a common unfolding-- of a cube-- ah, not quite-- and a regular tetrahedron. So it's actually a box not a cube.

One of the big open questions in this area is, can you find a common unfolding of a perfect cube and a regular tetrahedron that's open? Marty posed that question I think in 1998, in the good old days of the beginning of folding stuff. And this is almost an answer but it's not quite a cube, as a result. You could see-- I think it's pretty easy to see how to fold it into a box just with extending all the lines. And the dotted lines show how to fold into a regular tetrahedron. That's one example.

You can also fold-- this thing makes both a regular octahedron instead of a regular tetrahedron and some kind of tetrahedra. It's not regular. We call it a tetramonohedron, meaning each of the sides are the same. So it's just one type of side, but there's four of them. And I guess the red lines are folding into that, the green lines fold into a regular tetrahedron.

I feel like I'm getting forgetful. I don't even remember where this example comes from. But it's in the book, it's not attributed to anyone so I assume Joe invented it. But recently, like just this year, inspired a whole bunch of examples by these guys. I just saw the talk last week in China-- that's where I was traveling, it was a geometry conference. And Horiyama and Uehara talked about a bunch of different common unfoldings. This is, again, of a regular octahedron and a tetramonohedron. But a different example.

But they kept going. Said all right, let's do all the platonic solids. So this is a common unfolding of a cube and a tetramonohedron. Cube-- actually two of them. You can see how it makes a cube. That's obvious. But the tetramonohedron are the blue lines.

Then I have the octahedron, we have the icosahedron. I don't think they have a dodecahedron yet because all of these approaches are actually based on dissection techniques where you take a tiling of-- here are equilateral triangles for the icosahedron, and you take a tiling of some triangle that's going to make the tetramonohedron, and you align them so that things work out nicely. And that's not always well defined, but it often leads to good dissections, often hinged dissections. It's the way, for example, you show you can hinge dissect a rectangle into any other rectangle. You take tilings of different rectangles and overlay them appropriately.

Here, same approach, but with a dodecahedron, regular pentagons don't tile so it's very hard to use that approach. So this is as far as they've gotten. Here they've measured the proportions of the tetramonohedron to give you an idea. It's close to regular. But again, open question, can any two platonic solids-- do any two have a common unfolding? They proved in this paper, if you restrict one of them to be an edge unfolding and the other one can be a general unfolding then it's not possible. But if they're both general unfoldings, who knows. Here you see it's edge unfolding for the icosahedron, but general unfolding for the other.

Cool. But very recently, this is, like, hasn't even been released yet. This is the premier of the following, I'll call it conjecture. I think it's not yet approved or explicitly-- it's not certain that it's correct. But the claim is this gray shape, taken to the limit-- so this is the first iteration of the fractal, this is the second iteration of the fractal-- in the limit, will fold both to a regular tetrahedron and a cube. This would be an amazing breakthrough. Still, the conjecture is that it is not possible to go from a regular tetrahedron to a cube with a regular-- like a normal polygon with a finite number of sides. But with an infinite number of sides, it seems to be possible. Which is awesome.

This is-- I haven't met this guy yet. He's a professional puzzler in Japan, Toshihiro Shirakawa. So stay tuned for maybe a proof of that theorem, that claim.

The next topic-- this started back in '99, inspired by this question. We thought well, what about boxes? And this is a common unfolding of two different size boxes which is cool. And it's generated in a fairly intuitive way. You start with an unfolding of a box. This makes a one by one by three box if you think about it for a little while. Just wrap around-- this is like one top side, 1 by 3 rectangle and wrap around. And then you-- you draw the gluing-- here I've labeled which edges glue to which edges. And for every letter, like the two Gs here, I could imagine adding a triangle here and removing it from here. That will still fold to the same thing, right?

So I get to choose for each letter which guy goes in, which guy goes out. Then I get this shape as a result of that process. And if I do it right, when I rotate 45 degrees, I get a different orthogonal unfolding of some other box. So here, if I started with a root 2 by root 2 by three root 2 box, I get a 1 by 2 by 4 box. So this is a common unfolding of both. I guess here is like the center core of the 1 by 2 by 4 and you wrap around again. Or this is probably center core.

So that's pretty cool. This is an idea by Timothy Chan. Of course, natural question is, do I need to do this rotation trick or could I get away without it? And you can get away without it. This is an idea of Therese Biedl. This is my Waterloo days. We had an open problem session, kind of like the one we have in this class. It was the very first one that I ran. I ran it with Therese.

And she came up with this idea. This is a-- it's not obvious exactly how this works or where she came up with this. It was kind of magical. But this the same polygon, different creases, you make two different size boxes. Now they're going to be integers because they're orthogonal aligned on the grid. Pretty cool.

And for a long time, this was the only example of this that we had. There's still an open question which I give you A by C by-- A by B by C box and a D by E by F box, is this possible? When is this possible? But at least now-- oh, here's, sorry. There are two examples.

Of course, the surface areas have to match. So the sum of A plus B plus C equals D plus E plus F. I think that's right. Or no. It's more complicated. It's like A times B plus-- and so on. Pairwise products.

Now we know that there are actually infinitely many examples. This is another fairly new result by Uehara, again in 2008. This is-- Uehara translated our textbook to Japanese, by the way. Another connection. He does a lot of fun folding stuff. Just saw him last week.

So this is an example of taking a 1 by 5 by 2k box and a common unfolding of that with a 1 by 1 by 6k plus 2 box. Which has a very regular pattern. And this one, little more complicated to copy and paste but it makes 1 by 1 by 8k plus 11 box and a 1 by 3 by 4k plus 5 box. Which is pretty cool. I haven't studied these carefully, but there's two different ways to fold each.

So now we know there are infinitely many of these examples. It's still not known when it's possible, but also in this paper Uehara came up with a sort of random generation algorithm where he would take a box, randomly unfold it, try to fold it into other boxes. And he has now over 250,000 examples of this process. And I think these are maybe inspired by his random search. I'm not sure, but this is theory obviously.

One particularly fun example, this one actually tiles the plane. So it'd be efficient if you're cutting it out of sheet material. You can make a common unfolding of 1 by 2 by 5 box and a 1 by 1 by 8 box. I think this one he made into a puzzle with a paper manufacturer. I'm getting ahead of myself.

Big open question here that he asked is, are there-- is there a common unfolding of three boxes? That remains open. And he feels like he's very close to finding one with his random search but hasn't yet found one. It's like it almost folds into the third box. But it's strange how hard it is to find them. Maybe they don't exist. That would be surprising.

All right. Next example on this theme, common unfoldings, is this toy. So this is a 1 by 1 by 4 box, also known as the-- as a tetra cubes, four cubes joined edge to edge. And-- oh, I should have practiced. You can unfold it. This is a commercially produced toy. It's called Cubigami. And it's one net, one unfolding. But it can fold into all tetra cubes. So that was the tube and, let's see, what else can I make here?

I should have practiced. Interesting. It's fun. You can play with it later. Oh, that looks interesting. Almost. Not quite. It's almost-- it looks like a Tetris piece, right? Except I didn't quite cover it right at the end.

It's a good puzzle. Obviously. Maybe if I do it like-- that doesn't work. Wow. All right. You're going to have to play with it yourself. I lose. But you can make all of the four cube joinings except-- except for this one.

See, the problem with this example is it doesn't have the same surface areas the rest. And you have to conserve surface area. But if you think about all the other four cubes-- like I was trying to make this guy or this guy-- it's like Tetris pieces-- but also things like-- I can draw better than I can solve the puzzle apparently. Like this. This is the one non-planar one. All right?

So you've got all the Tetris pieces except this guy. It's the wrong surface area. So, in general, it's all the tree shape ones. No cycles. Can make all of them. This is an example found by Donald Knuth, famous computer scientist. He sort of accidentally at a cocktail party heard about this problem from George Miller who's a toy designer-- or puzzle designer. And so then Knuth went back and he wrote a program to numerate all unfoldings of these things and found there are a whole bunch of common unfoldings. But it's found by exhaustive search.

This was the one that sort of had the smallest bounding box, I believe, in terms of area. And so it's a nice compact thing. And then there are various manufacturing's of it. This is the latest one that's sort of relatively easy to build I guess. So, play with that later.

That's four cubes. What about five cubes? Well, that was the subject of a paper also presented last week in China. And it turns out, there is no common unfolding of all 24 pentacubes. That's five cubes joined together. But there is a common unfolding of 23 out of 24 of them. So this is the paper. There was a bunch of authors.

There were basically three teams that worked on this, two of which were writing code. I was on one of the coding teams. And the third one was trying to think about the problem using mathematics. And for a long time, the thinkers were leading. But then, of course, eventually the computers win because after a couple days of just solid computation on I don't know now how many cores with each of the groups, we numerated all these unfoldings.

And so now there's, I think, over one trillion unfoldings of the 1 by 1 by 5 tube. But one trillions not that bad. Now, we could not do this for six. But for five it wasn't so bad. And this is-- there's a bunch of unfoldings of 23 out of 24. But that's the max you can get. Little disappointing.

**AUDIENCE:** Which pentacube lost?

**PROFESSOR:** I don't know which pentacube lost. I should find out. That's a good question. But I do have some other answers. So in some sense, the hard ones I think are the non-planar guys. Like this guy.

So they're planar if all the cubes lie in a thickened plane, thickened by one. This guy will fold into all the non-planar cubes and also fold into most of the pentacubes. There are only 24 total so it's only missing two. And this'll fold into all the non-planar guys. And what's cool is this one is unique. There's only one common unfolding of those guys.

**AUDIENCE:** How many non-planar [INAUDIBLE]?

**PROFESSOR:** How many non-planar are there? I don't remember. Maybe 10. I think it's a minority. But there are still a bunch of them. And I think this would make a good puzzle. If you're going to manufacture-- if you guys want to do a start up, here you go. Freely available.

If you want to do the planar pentacubes there are a lot of common unfoldings of them. This is one of the fairly compact ones. And this one, I have an animation of. Or, actually, not this particular unfolding, but this concept. And that is here. Yes, I trust myself.

So here is unfolding and re-folding of that particular shape. And then I hit Space bar and it's the same unfolding but a different polycube. And this is going to go through all of the flat ones. And that animation is done heuristically. I don't have any theory that that's always possible. But always keeping the same unfolding, just changing where the creases are. That's so cool.

It's funny, actually. The person who implemented this was the guy who was thinking, not the coder-- coder teams. I guess he had free time. This is by [? Karim ?] [INAUDIBLE]. I guess you can count them and then subtract from 24 and you'll get how many non-planar ones there are. But I've lost track of where we are. Always same unfolding. This is a fairly nice, simple unfolding. It's mostly one dimensional.

All right. You get the idea, right? Pretty fun. All right. So, right. Do I have anymore? I think that was the end of my slides. I do have one more example in the notes which is, if you just want to make the flat guys-- so just the two dimensional examples-- so we know we could do that with five, but you can also do with six. We wrote another program that looks at a very special kind of unfolding. And in that special class, it's possible to fold all planar six cubes-- hexacubes. But that class does not work for septacubes-- seven cubes.

Open question, is there a common unfolding of all planar seven cubes or all non-planar seven cubes? Or how many can you make? There it was beyond exhaustion, and we have to actually think. Unless you come up with a good class and then you can exhaust it in that class.

One nice open question here is, if you go to large N, are there some two polycubes that have no common unfolding? At the moment, it could be every pair of polycubes has a common unfolding. Certainly there are-- for example, in this class there are four polycubes that do not have a common unfolding. I think we're not sure about three. For size five.

Of course, we don't even know whether there's any polycube that has no unfolding. I think there should be one. But at the moment, it could be every polycube has an unfolding. But to make it a little harder, what about two polycubes at once you want a common unfolding? All these questions are open.

All right. This is common unfolding. We're going to stick with unfolding stuff but change around the problem a little bit. So let me ask you a question. Can have a vote for those who haven't read the notes already and know the answer. So let's say, orthogonal polyhedron. We talk a lot about orthogonal polyhedra. See all this polycube stuff is in that genre. And so way back when we talked about grid unfolding of orthogonal polyhedra. It's not known whether that's possible, but we have epsilon unfolding where you cut exponentially many times and so on.

But how do you define orthogonal polyhedra? There are two natural definitions. One is that the angles are right. Right angles, let's say, in the faces. The other natural definition is that you look at two faces and the angles between those should be right. So I guess right dihedral angles. Right face angles or right dihedral angles. The question is, are these the same thing? Who thinks yes? Who thinks no?

Yeah. I kind of set it up. Everyone thinks no. They're not the same. But for genus zero polyhedra, they are the same. That's maybe the surprising thing. This is a genus eight polyhedron. No, seven. It's always a little tricky to measure genus. If it just had one triangular face that would be-- well, anyway, that's genus seven. I'm not going to try to convince you.

This is an unfolding of the thing. So you can see, in the unfolding, it looks like it's an orthogonal polyhedron. All the angles are right. It looks perfect. And yet, it folds into this non-orthogonal thing. Kind of a thickened octahedron. It's kind of critical that it has a genus more than zero.

In this paper-- this is the original paper that considered this problem-- there's a lower bound of one. The genus 0 is not enough. Then we came along-- I don't have the author list here. It's with the Waterloo gang, it was Therese Biedl and others. And this is a smaller genus, a genus six polyhedron. This guy. So it's almost the same example turned inside out. So this like to half octahedra and then joined in this cubicle frame and then you put caps on the cubicle frame to make there not be holes out there. And then you have slightly smaller genus because you can go around the backside.

So that's the best known, is a genus six example. We proved that you need genus at least three in order for such an example to exist. I have a little sketch of the proof in the notes if you want to read it. It's a fun little boot strapping argument but I would-- I'm going to go on to other fun topics here rather than prove this. Is that a good plan? I think that's good plan.

OK. So this is a fun kind of-- unfoldings can be orthogonal while the three dimensional versions are not orthogonal. All right. Next topic. I'm going to go back to Alexandrov's theorem. Turns out there is a smooth version of that. So now we're going to segue a little bit into smooth foldings and smooth unfoldings. So instead of thinking of polyhedron objects-- polygons and convex polyhedra-- I want to think about smooth bodies.

So, for example, suppose I take a smooth convex shape like an ellipse or something and I picked two perimeter antipodes and I do perimeter having gluing where this is defines-- as I walk along the perimeter in both directions it-- I obviously didn't do it perfectly, they should meet at exactly the same time. This defines a gluing. For every point there is a corresponding point it gets glued to. This is a way to make what is locally like a convex surface because at any point, at most 360 degrees of material is glued there because this thing was convex.

And the same way as for convex polygons, does this make a convex something in three dimensions? It's not going to make a polyhedron because it has this curve, but indeed it does make a convex smooth surface. So every convex metric that's topologically a sphere is realized by a unique convex surface.

So Alexandrov's theorem was identical to this statement except it also had polyhedral up here, that there were finite number of points of nonzero curvature. Now we have infinitely many points of slightly positive curvature because, again, this is supposed to sum to 4 pi. That's still true. But it's now an integral instead of a sum. So each of these things sort of has an infinitesimal curvature and if you integrate them it adds up to 4 pi. You can check that.

So this will make something. I encourage you to try this out, again, with tape. You're going to approximate it because tape is going to be polygonal. But it works. It will make something. I particularly-- I should mention the idea of this proof is just to take limits of regular Alexandrov. Take closer and closer polyhedral approximations. In the polyhedral world we can use that induction that we talked about. In the smooth case, you can't, but the smooth case is the limit of the polyhedral case so it works out. This is proved by a student of Alexandrov Pogorelov in the '70s, whereas he proved his theorem in the '40s or '50s.

One really fun thing you could do with this theorem is, instead of using just one convex shape, you could use two convex shapes and zip them to each other. And this is called a D-form. It was invented by an artist, Tony Wills. So here I've taken two-- in this case, two identical convect shapes-- they don't have to be identical-- picked two points to glue them to each other and then you just zip around the perimeter.

And again, because you're always gluing convex angles to convex angles-- the smooth convex angles-- but they're still less than 180, so you glue them together, it's less than 360. And in this case, it's-- we could actually figure out and think about what 3D shape you get so we could build that in Mathematica. But in general, you get some weird convex shape.

For a while there were some examples of this but no one knew exactly what they looked like. Then, I think it was a project in this class or it came out of the problem session in this class three years ago, Greg Price and I proved that these things are nice in the sense that the only place you get creases is along the seam, the pink stuff. Both for-- these are called D-forms, these have another-- I think these are pita-forms, if I recall, where you take one convex shape instead of two.

Here you only get-- you can get the creases along the seam and one extra crease, if I recall correctly here. And here you don't get any extra creases. It's always smooth except at the seam.

And also, if you just take the seam-- the pink part-- and take the convex hull of that thing, you get the same thing. So it doesn't go outside. It's really tightly wrapped around wherever the seam happens to go. Remember, this thing is unique by Pogorelov's theorem. So it's kind of nice to know that structurally these are pretty nice and smooth. If your original shapes are smooth convex bodies, you will get a nice, smooth shape at the end that only has creases at the seams. Cool. Again, please try to it at home. It's lots of fun.

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** Let's see. This point should glue to here. I mean, it's perfectly symmetric.

**AUDIENCE:** I'm looking at the 3D form.

**PROFESSOR:** The 3D form. So one of the two shapes is here and the other one is here and goes around the back to there. And it's, again, symmetric. This is a nice example because we have those parallel lines. Usually you wouldn't have that. And they're harder to draw, is the challenge. Here, obviously, we're doing sort of a polygonal approximation. You see all the lines.

All right. This is a smooth version of folding-- a smooth version of Alexandrov's theorem. We can also think about smooth versions of unfolding. So here there is exactly one paper by Nadia Benbernou whose in the front row and Heather--

**AUDIENCE:** Patricia.

**PROFESSOR:** Patricia Khan and Joe O'Rourke. So this is a prismatoid. It's a little hard to see the bottom face but there's a convex polygon on the bottom, a parallel convex polygon on the top. You take the convex hull-- sorry, they're not polygons. They're smooth convex bodies. So you've got one in the floor plane, one in a parallel plane up top. You take the convex hull, you get this nice smooth thing around the outside.

Now, we have to generalize our notion of unfolding because normally with unfolding this would take an infinite number of cuts to-- we have to cut everywhere there's curvature. Which means we're going to have to cut along all those black lines. Which means we slice the thing into very infinitesimal pieces. But if you think of it as a limit of regular polyhedral things-- here I have a little example.

Let's think about the limit in a simpler example to start with. So imagine you want to take a pyramid-- so the top poly-- the top convex shape is just a point. I guess it looks like a cone but down here it could be anything, could be an ellipse or any convex shape. Well, we can think of this as a polyhedral approximation. So, for example, we take a hexagon on the bottom, we want to unfold that. And in particular, we thought about these-- these are the pyramids-- and we-- one of the unfoldings of them is the volcano unfolding if you remember back to the unfolding lecture. Looks like that. All these triangles just fold out, get all those parts. That never overlaps, it's sort of trivial.

You take the limit of that. In the limit, the floor becomes whatever the convex shape. Maybe-- let's think about the simple case where it's a disk. And in the limit you're getting lots of really tiny triangles. You take that all the way to limit, what you're getting is actually a concentric circle around there. So really, these become segments.

Now, what's funny about this limit is, normally-- before you get to the end, you see there's these big gaps. A lot of the area's missing. These things are getting infinitesimally small and yet they have to spread over this big range. So most of this area is actually absent. When you take the limit you can't really see that anymore. It just becomes filled in. And so in this unfolding, the area of the green stuff, if you just took that area measured on the floor, it's going to be larger than the area in the pink. So this is not an area preserving unfolding which makes it a little weird.

But it's still well defined. I mean, you wouldn't actually-- well, for example-- we're going to get to this in a moment-- if you actually took material like this out of, say, tin foil, something that can crinkle, then you should be able to crinkle it into this shape. I'll talk about how to prove that. That's actually most of the rest of the lecture, is about that kind of notion of folding. It's more like origami, though, than unfolding because you're sort of over covering stuff.

And here-- actually, here the hard part is not so much to show these ribs don't intersect, although they don't and it's not totally obvious because here they're kind of going towards each other. But they won't actually intersect in the unfolding. One of the tricky parts is to show that the top face actually can fit somewhere, so you actually get an unfolding of this thing.

And I forget, this is choosing some kind of extreme rib and putting it out there. It's a nice line of separation if I recall. This is funny because here we have a limit of some process. Now, for pyramids we know the volcano unfolding works. For prismatoids, regular prismatoids are when you take some convex polygon and some other parallel convex polygon, take the convex hull.

We don't know whether these have edge unfoldings. Still an open problem. And yet, for the smooth case, we do. Kind of magical. I know why that's true. It's somehow the edge constraint is harder. If you're not just worried about edge unfoldings, we do know how to unfold those obviously. You can do star unfolding, source unfolding. Here, edge unfolding-- well, we have to cut along all those infinitely many edges. That makes it easier, much smoother. But it's funny when the smooth case is actually easier than the discrete case.

All right. The next problem I want to talk about is chocolate. I have here a bunch of spherical chocolates. The most-- apparently the only perfectly spherical chocolate, at least according to the advertisement, is this thing called the Mozart Kugel. Now, Mozart Kugel-- how many people have eaten them before. Just a couple. Good stuff. It's, I guess, more a European thing.

It's from Austria. It's invented in early 1900s, I forget exactly when. It was a big thing at the time because making chocolate perfectly spherical is not easy. It's like, how do you hold it and not get little indentations? This thing is perfectly spherical. Here I'll show you. But, really, the question I wonder, because as an origamist you wonder, how do you wrap this thing? Oh, I ripped it. I'll try to do this more carefully.

It's not how you usually eat chocolate, is it? Open very carefully. It's OK. I have more in case I fail. More chocolate. Not enough for you, I'm afraid. So ignore the chocolate. It's not the Droid you're looking for. It's not the chocolate.

So it's a rectangle. I didn't do a perfect job. There's a few little tears there. But they fold-- Mirabell. Mirabell makes the most Mozart Kugel in the world. Mozart Kugel is very cool because it's built in-- built up in concentric spheres. You have the marzipan layer, then the milk chocolate layer, then the nougat layer, then the dark chocolate layer, if I recall. Each brand has a different recipe, but they're always spherical on the outside. It's kind of like glassblowing actually. It's kind of fun.

And each company has a slightly different way-- well, I've studied only two main companies. Mirabell makes the most of them but they are not the original. They are called the echta which is--

**AUDIENCE:** The real.

**PROFESSOR:** The real. The authentic. But-- now, these are hard to come by. These are brought to me by my student, [INAUDIBLE], some of you know. Architecture.

This is the-- oh, it says in English. It's kind of easy. The original Salzburger Mozart Kugel. Mozart was-- lived in Salzburg, I guess. And so this is named after him. Oh, 1884. Wow. So that's when they started.

Now, these guys-- I think I've only had like one of these in my life. I'm really excited. The Mirabell's are OK, but this stuff? Wow. Oh, it's so much easier to unfold. I like-- this is, I think, probably handmade. Much more handmade. In fact, you can kind of tell it's not perfectly spherical. It has a little nub at the top. So this is probably more how they were originally made.

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** What, are you getting hungry? Now, Furst-- Furst is the company who made the first Mozart Kugel-- they use squares. I claim they're a little bit better. But interesting question is, which uses more material? Something to think about. Meanwhile-- I won't torture you more by eating.

All right. So this is a Mozart Kugel problem. This paper started when-- I think we are in New Orleans. Some part of the gang was in New Orleans and there was this deadline the next day and it was for this workshop on computational geometry-- the European workshop-- and it happened to be in Austria that year-- that coming year. We're like, what paper could we write? And we're sitting there in the little cafe eating these beignets of doughnuts and thinking, what could we write a paper about?

It's like, how about food? And what does Austria have? Well, [SPEAKING GERMAN] Well, how about Mozart Kugel? So-- this is how research happens. But it's really cool. It's really cool because this is not your-- this is not the origami you grew up with in this class. Because you can't fold a sphere out of a square paper. And yet, this was a square folded into a sphere. Somehow that was done using a finite amount of effort.

Whereas if you use regular paper to fold something that has positive curvature everywhere, you need infinitely many creases. And yet this is fairly practical to do. This is hand wrapped. In theory, I could re-wrap this. I've never actually tried. But something like this. It works pretty well. I even covered the tip.

How did I do that in finite effort? Well, of course it's not a perfect sphere. But we need to model this mathematically. It's annoying to take approximations. I would like a model of folding that actually does make it possible to fold a sphere. So this is the idea of contractive folding.

We haven't been super rigorous about defining origami, but I always say you take a piece of paper, you can't stretch the paper, you can't tear the paper, and you can't self-intersect. We're going to relax the you can't stretch the paper. You still can't stretch the paper, but now the paper can contract. So it can't get longer but it can get shorter. It's like cables from tensegrity theory.

So you take any pair of points. There's some distance as measured on the piece of paper. When you fold this thing into your awesome sphere, you take the two points, you measure the distance along the surface of the piece of paper. Now, I'll-- normally with origami, these two distances should be equal. That's isometry. That's regular origami.

But if I say that this one-- so write equal that way. If this one can be smaller, then this is contractive. And that is paper that can shrink but not stretch. So pairwise distances don't increase. So they could stay the same. We could still do regular origami. But we can also contract. And then it is possible to fold a sphere.

Now, intuitively, what's going on is, when you fold out of tinfoil, you can crinkle. And that lets you not have to worry about-- I mean, this is pretty good approximation to a sphere especially when I have a good base underneath to approximate against. I'm doing lots of little tiny creases in order to approximate that contractive mapping, contractive folding. So, conveniently, there's a theorem that tells us this is always possible.

I'll tell you the theorem. This is another theorem by Burago and Zalgallar who proved the other theorem that if you have any polyhedral metric you can turn it into some polyhedron, not necessarily convex. So they solved that problem plus they solved this problem. If you take a contractive-- they only solved the smooth case. I imagine this works for piecewise smooth. You take a contractive C2 immersion-- it's just some fancy word for folding, it has some extra constraints but let's not worry about it.

So C2 means continuous up to the second derivative. So nice and smooth. Spheres are valid, for example. Then any one of those things has a C0-- I'll just call it polyhedral approximation. And this is approximation in the Hausdorff sense meaning you want to fold the sphere. If you just thicken that sphere by some epsilon, then you can sort of stay within that epsilon thickened sphere and do some kind of wiggling. You'll be polyhedral and you'll only have finitely many creases. So there-- if you have a nice, smooth contracted folded state, you could turn it into a regular origami isometry folded state that's arbitrarily close to the sphere. So for any epsilon.

So that's a convenient theorem from 1996. So that's the model. Now we can think about, how do I optimally wrap a sphere? Now, there are many possible goals you might want to optimize in wrapping a sphere. Presumably, you want your shape to tile the plane. So you can cut out a whole bunch of these out of one sheet of tinfoil. Obviously Mirabell likes rectangles, Furst likes squares. Are those the right things to do? What about triangles? What about other shapes? These are the sorts of questions we consider.

We don't have a complete answer to this question. But we have a lot of fun-- fun things to do. What should I tell you about first? I'm getting hungry. Maybe I'll have the rest of that Mirabell. It's just sitting there. Half a Mozart Kugel, can't have that. All right.

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** Yes. Question?

**AUDIENCE:** I was wondering--

**PROFESSOR:** You can have any.

**AUDIENCE:** Because of the fact that you've got this contracting that occurs, I was wondering-- it's obvious that because of contracting you need more than 4 pi r surface area of paper to wrap a sphere-- a one unit sphere. I'm wondering, do you know exactly how much paper do you require due to the fact that [INAUDIBLE]?

**PROFESSOR:** Good question. How much paper do you require? Area is not being preserved here. Just like the continuous unfolding stuff, you don't preserve area. This is really the same thing. I think you can prove this is a contractive map. So if you had this as a piece of paper, you really could fold that smooth pyramid.

But you lose material when you do a contractive mapping. And in the inverse, when you're unfolding, you gain material. How much material do I have to gain? Think about it. You should be able to answer that question. I'm going to tell you about some stuff while you think about it.

That was the original question we started with. What's the minimum area wrapping of a unit sphere? The surface area of a sphere is 4 pi, FYI. How much more do you need?

All right. To describe these unfoldings, I guess they are, or wrappings, whichever way you're thinking about going, it's useful to have some structure, kind of a backbone to the unfolding. So we came up with this idea of stretched path. And you can show that any kind of optimal wrapping will have at least one stretch path. And our idea is actually-- let me tell you what a stretch path is.

This is a, let's say, a path between two vertices whose length does not decrease. So it actually is preserved in the folding. So I have my sphere, I have two points, take the shortest path between them. That shortest path is exactly what it is in the unfolding, exactly the same length. So that's what we call a stretch path. There's no crinkling along that path. There maybe crinkling on either side of the path, but not on it.

Our idea is-- OK, that gives you some kind of structure. Our idea is to completely cover the surface of the sphere with stretched paths. There's a lot of ways you can do that. Some of them will be valid, some of them won't be. The simplest one is the source unfolding. Or I guess you could call it the source wrapping. Wrapping somehow means contractive I guess.

Which is, you take your sphere, take some point x, you take all the shortest paths around x-- it's a little tricky to draw. In other words, the Voronoi diagram at that point. Just like we did with source unfolding. And you call all of these edges, all those straight lines-- geodesics, whatever-- they go to this opposite south pole here. All of those lines you call stretch paths. That determines an unfolding. And it happens to be contractive.

The unfolding will be a big disk because you'll have the center-- call this x. This is 3D, this is 2D, in case it wasn't clear. They both look the same. But now all those paths are straight. And it's clear what the lengths of these paths are. Each one of them is half the circumference. Circumference? Is that what you call it? Half of an equator of the sphere. Which for a unit sphere is pi. So this is radius pi, which means the area of this thing is pi r squared. Sorry, the area is pi cubed.

Interesting. But pi cubed, you can compute, is more than 4pi. Pi squared is almost-- is over nine. It's over nine times pi. We want four times pi. This is an example of not preserving area.

What's happening is there's lots of crinkling between these lines. Now, there are infinitely many of these lines. Every point is covered by one of these stretched paths. But again, when I unfold them, if I did a limiting approximation, there'd be lots of holes here. But in the limit there are no holes and that's where I waste material and that's where I get contraction. Make sense?

I actually brought a sphere, a couple spheres. I brought some chocolate spheres, but slightly bigger sphere so we can think about this. In this case, we're making stretch paths all around from the north pole down to the south pole and kind of cutting at the south pole. And that's it. All right.

Has anyone solved the problem of what the minimum area wrapping of the sphere is? Yeah? You solved it? Janine. Or an idea.

**AUDIENCE:** I think it's 4pi plus epsilon.

**PROFESSOR:** 4pi plus epsilon is correct. How do you want to do it? Exactly.

**AUDIENCE:** An apple core-- or peeling.

**PROFESSOR:** Apple peeling, yes. Or I prefer to think of it as lecturer-- I don't know which one-- the one about strip folding.

**AUDIENCE:** Oh, yeah.

**PROFESSOR:** Yes. Exactly. So we had this result about strip folding if you remember way back in the folding anything lecture. We talked about this and then orgamizer. This was the inefficient method. But if you started with a very long rectangle of paper, we could show that the area of that piece of paper is arbitrary-- could be made arbitrarily close to the area of your polyhedron.

So that area plus some epsilon. And you can make epsilon as small as you wanted just by making this polygon longer. If you take a polyhedral approximation of a sphere and apply this theorem, you'll get very close to the surface area of that approximation which is very close to the surface area of the sphere, which is 4pi. And then, if you take an outer approximation, then you can just sort of crinkle all those vertices down against the sphere. And you'll do-- it's maybe not totally obvious, but it's not that hard. You get a contractive map that actually wraps the sphere using not much more surface area than what you had.

The same as what you had, which was not much more than the sphere. So you can get 4pi plus epsilon. So it turns out, minimizing the area of your wrapping is not that interesting a question by itself because there's kind of this cheating solution. Obviously, you don't want to manufacture that. I don't think. Maybe she built a robot, did the inverse orange peeling, it would be pretty cool.

But if you're trying to sell this idea to confectioners around the world, I wouldn't recommend attempting strip folding. So we would like something that's reasonably nice but-- yeah. OK.

**AUDIENCE:** So if you're trying sell wrapper companies [INAUDIBLE] square or rectangular amounts would these be 4pi squared? Because you're basically producing something that maybe would expand the circumference and--

**PROFESSOR:** Well, OK, an interesting question is, what is the surface area of these wrappings? These actually have pretty special properties, the rectangle and the-- I'm getting hungry-- and the square. Have you thought about which of these has more surface area? The rectangle has more? Who thinks the rectangle has more? Who thinks the square has more? Who thinks they're the same?

Seed you with that idea. Well, they're actually the same. I'll explain why in a little bit. It's crazy, I know. I prefer the square because it has less perimeter.

There are obviously many things you might not like about this strip folding. It has high aspect ratio, but in particular, one way to characterize why this is nasty, it says high perimeter. Perimeter is annoying because-- for a few reasons. Intuitively, if you started with the sphere and you wanted to cut it open, then you need a lot of cutting because the perimeter is the amount that you cut. Or it's twice that. So conversely, when you're trying to wrap a surface, if you have high perimeter, there's a lot to worry about, lots of things that have to match up. So you've got to be very careful in your crinkling. Whereas I really didn't have to be very careful wrapping with the square because it has low perimeter.

Another sense in which perimeter is annoying is you're going to have some error. So in fact, you can't just perfectly wrap a sphere. Like, if I took a disk of exactly this size, I probably couldn't wrap a sphere with it because I'm going to lose just some epsilon and then there'll be a little point around the south pole that's missing. If you're really going to build confectionery-- build confectionery-- if you're really going to wrap confectionery, you probably make this a little bit bigger. How much area do you lose when you make your thing a little bit bigger? Well, it's roughly the perimeter times epsilon. It's a little bit different from that, but basically perimeter times epsilon.

So if you minimize perimeter, then your area wastage from double coverage will go down. So really what I want is something that's low area and low perimeter. That's our new goal. And that's why I like Furst better than Mirabell, not just because they taste better. I really should verify that they taste better. Oh, yeah. Now, they're very different insides. Oh, but it's so good. Oh, much better chocolate. All right.

Maybe I should have a lottery for eating the rest. Petal wrapping. This is going to be another way to put stretch paths on the sphere and get an unfolding-- get a contractive wrapping from that. Which actually corresponds to this toy. I don't know if you ever played with this. The Frisbee that's also a ball.

These-- you see those cuts? Those are the shortest paths from the north pole to the south pole. But we're not-- before we were sort of using-- kind of slicing along but also preserving-- that's kind of confusing-- all of them, infinitely many. Now I want to do with finitely many. Here I only have six. Six? Yes, six.

Doesn't have to be six. Could be any number k. So I'm going to look at k stretch paths from north pole to south pole. So I have my sphere, here is my north pole. I do some constant number. Here I've done five. This is equally spaced angular around the north pole, they all go to the south pole on the bottom. Something like that. It's going to start getting messy.

These are going to be stretch paths. I have to fill these orange slices in between them. How should I fill them? Here's an idea. It's kind of like the Voronoi diagram. Voronoi diagram is you grow from many objects simultaneously. With this sphere wrapping we grew from a single point in all directions. That was it.

Now, I've drawn these paths. What if I grew from all of these finite stretch paths at once? What I would get, these would at some point converge. They would converge at the angular bisector here. Along the way, what's happening is each of these points is kind of tracing along a straight line, so to speak, a great circular arc until it reaches that point. I'm going to make these stretched paths, and these are also stretched pads, and basically cutting along the dotted line.

OK. I won't draw all of them. But you get the idea. When you unfold this thing, you have these kind of wish bones. These guys meet sort of perpendicularly. So when you unfold you preserve those angles. Remember these are stretched paths. These lengths are preserved. The stuff in between, it's really a curved thing and it gets slightly larger in this thing. We call this thing a petal.

But this is an unfolding of this sort of one fifth of the sphere. This is the center line and these are the two sides. So that's the idea of petal wrapping.

Now, there's different ways to join these petals together. One idea is join them all at the north pole. So then what you get-- it's a little hard to imagine that these things don't intersect each other, especially when I've drawn them so fat. For five, it's going to look something like this. I'm drawing it perfectly. They won't actually intersect in the center because the total amount of material at the center is only 360 degrees. And this will actually have-- the tangents will actually match up here. So they will perfectly mean in the center without self-intersection.

So it's only sort of a vertex unfolding. Because they're only connected at the single point. But that's maybe necessary. I don't know. And again, these are the stretched paths. It's a neat way to think about unfolding.

You can draw these-- ignore the yellow for now. That's if you might want to actually cut these out. So we have the three petal unfolding, the four petal unfolding, five petal unfolding, six petal, you can go arbitrarily high. In the limit, the area of the blue stuff will converge to 4pi. This is another way to get arbitrarily close. Because the smaller these things are, the less you're distorting.

So this is two families of wrappings. We call these-- the blue stuff is the petal unfolding, the yellow is the convex hull of the petal unfoldings. Not quite, actually. So in the case of k equals 3, we could think about a convex hull, but what I've drawn here is the smallest enclosing equilateral triangle. Now this is the first class of unfoldings we came up with after the obvious sphere-- or circular one which had pi cubed over 9pi.

One thing we wondered is that triangle, is it better than the Mozart Kugel wrappings? Or is it worse? One way or the other, presumably. Or maybe it's the same again. Who knows. Let's see.

Yeah. Let me tell you, at this point-- so I need more techniques to cover both of those unfoldings, but that square is the first wrapping. This one. I'll have to unfold another one just so you can see. All right. So you see-- I mean, I did a few times so maybe you already know. Here I have four corners, four right angles. They meet at the top. So that is-- that's where the four corners of the square which are also the four corners of the petal come together. And then it's just everything-- these lines are stretched paths. There's no distortion around them, they just kind of unfold. The stuff in the middle is-- it's like undressing here. It's so tantalizing.

The stuff in the middle is getting contracted. So if-- I don't know if you could see, maybe in the video you'll be able to see. Look closely at here. Along this path there isn't so much crinkling. In between there's a lot more crinkling. It's like here there's a lot of crinkling. When I unfold it, a lot more stuff happens in between the four top paths. OK?

Now, they wanted to cut very simple shapes. In particular, shapes that tiled the plane. So they ended up using squares instead of the sort of optimal with that stretched paths which are those four petals. So they used the yellow, whereas you could just get away with the blue. Of course, you want to add a little bit of material in the center so it doesn't fall apart. And you need to thicken the edges a little bit. So they care about that little yellow minus blue savings? I'm not sure.

But what about the triangle? We spend a lot of time computing these shapes. It's some weird trigonometry. It's in the notes. Arc sine of sine over square root of 1 over sine squared minus cos squared of various parameters is the description of one of these curves. So we did all this computation and then we integrated it to compute what these areas were and then-- or figure out exactly what these shapes were, take the tangents, say, well, is the smallest enclosing triangle better or worse than the square? It turns out it's better by 0.1%.

Now, I think-- millions of Mozart Kugel are made every year. 0.1% could equate to huge savings. Dollars. But just think of the material wastage that Furst is committing by using the square. If they used the triangle instead, I mean, this could solve global warming. This could solve chocolate melting. We actually talk about global warming in the paper as a joke. It's a very funny paper. Very silly paper, I should say.

Right. Let me tell you another wrapping. Chalk. The comb wrapping. Comb wrapping, we're going to use the same idea. Again-- do I really want to draw this again? I'm going to draw those petals, but instead of connecting them at the north pole, we're going to connect them along the equator. This is actually what happens-- oh, no. This is not-- this connects them at the north pole I guess, and it folds them in half. Whatever.

But I have the equator nicely denoted here. So I want to keep the equator as a new stretched path. I want that to unfold straight. But then attached to it perpendicularly are all these petals which unfold. So the result looks like this. Here's the equatorial stretched path and here are the petals. Petals are exactly the same shapes as before, so we still have stretched pads perpendicular like that. But it's a new way to cover every point on the sphere with stretched paths.

It leads to the blue unfoldings. You can take the smallest enclosing rectangle and you get the yellow. What do I want to say about these? They're the same area, of course. They look a lot like these old cartograms, cartographs-- whatever you call them-- unfoldings of the earth. I thought I brought an earth. Maybe we didn't bring the earth. A globe, that is.

You've seen pictures maybe like this. They're not unfolded in the same way. They don't have stretched paths that are perpendicular. They use some other projection, so their petals are a different shape. And I'm not sure that they're contractive-- or inverse contractive maps. But same spirit anyway. Let's see.

This one is the Mirabell unfolding. Should I unfold another just to illustrate? Going through chocolate like crazy here. This is the really hard one to unfold. So here's one corner-- a little hard to see. Where's the equator going to be? I think the equator is around here. That make sense? No. Actually it should be perpendicular to that. Around here. This is a little harder to see.

Here's a corner, actually. Tucked inside. I've been really curious what kind of robots they have to do this. All right. Here's the equator. Now I can see it. So this-- the long edge of the rectangle is wrapped around the equator. See it come off? And then the corners are wrapped up against-- so there's two-- well, yeah, these guys go on one side, these guys go on the other. That's the short edge of the rectangle.

Now we can actually compute how big these things are, right? So the-- where's the-- I have a page about this. Not much. The vertical length, that is one half equator. So that's pi. The horizontal length is 2pi. So this is Mirabell rectangle, pi by 2pi. And we talked about the Furst wrapping which, let's see, this length-- so this goes like this. So this is 2pi, right? The diagonal is 2pi. So that means this edge length is root 2pi. By root 2pi.

And magically-- I don't really have a great reason why this should be the case but-- well, maybe I do-- but these have the same area. Both 2pi squared. Which is like 6pi. So better than 9ish pi, but still not quite as good as 4ish pi. I have some exact numbers. More exact numbers?

Oh, I have the perimeters. So the first wrapping, the perimeter is 5.7 times pi. And the Mirabell wrapping is 6 times pi. So reasonable savings in perimeter. 2pi squared. The equilateral triangle wrapping-- that one in the top left-- instead of 2pi squared, it has area 1.9983pi squared. That's the 0.1% improvement. Amazing.

This was a surprise. We thought it'd be a fair amount better or worse or the same or something. But almost the same. It's not what we expected. Let's see. I want to go to packing.

What if you want to do better than those simple convex shapes? Yeah, equilateral triangles and squares and rectangles pack. But what if I tried to-- what if I allowed non-convex shapes? I still want to tile the plane, so I'm going to lose some material for that. I could add some of this material to the blue guy, some to the purple, some to the red. You can compute the area of this thing and it does a lot better than the equilateral triangle even. So now we're getting down there. The perimeter goes up a little bit, but not by much.

Packing three petals gives 1.6 times pi squared. Getting better. By the way, 4pi-- because I'm going to speak in pi squareds now-- 4pi is about 1.27pi squared. OK. So what we want is 1.27pi squared. At this point, we're at 1.6033 times pi squared.

The comb does better. In the limit, you don't lose zero here in the limit, but you lose quite little. In the limit you get 1.333 times pi squared. Versus 1.27. That's pretty good. I think this is-- this is what we should manufacture. Maybe. Of course, the perimeter goes to infinity also. Slight problem.

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** You don't lose zero in the limit. For awhile we thought we did. You'll have to think about it. It's not obvious. Really, we have this two dimensional space. It's called the Pareto curve or-- it's a plot of perimeter on the x-axis versus area on the y-axis. We want both to be small.

So two-source wrapping, for example, is pretty good in both metrics. That's where you take the Voronoi diagram, both the north and south pole. So you actually have two disks-- one for the southern hemisphere, one for the northern hemisphere-- you get two disks and you just attach them side by side. Although-- yeah. Let's see. What's this one?

There's the Furst wrapping. So they're really good in terms of perimeter. In fact, one of the best perimeters we know. It's an open problem how far left you can go. The stars are convex hull of petal wrappings. Because the square was a convex hull of a four petal wrapping. This is the convex of the three petal wrapping. Now, that doesn't actually tile, but it has the smallest perimeter we now.

So it's where you take the three petals, but instead of completing it to a triangle, you actually make the sharp turn around the petal. So we have three petals like this. And so the convex hull is some tangent here. And then you follow the petal for awhile. And then you take a tangent, and then you follow the petal for a while, and then you take a tangent.

So that's the smallest perimeter unfolding of a sphere that we know. Open problem. Can you do better? Can you get arbitrarily small? Probably not. Some isoperimetry problem.

You can get arbitrarily low area, arbitrarily close to the optimal of 4pi. That's like the strip wrapping. Let's see. So that's one limit. That corresponds to the limit of petal or comb wrappings. Petal and comb wrappings, if you don't take any hulls or anything, they are the same area and the same perimeter. So I think out of that you should be able to see why, in the special case of the square and the rectangle, you get the same stuff. There's a bijection between all the parts. Not totally obvious. You have to dissect a little bit.

Mirabell wrappings, they're not so good. What are those? The convex hull of comb wrappings. So they're way out here. Clearly Furst is better. Here's the equilateral triangle, which you can't tell, but it's slightly below that dashed line. And pretty good perimeter but not quite as good as Furst. So actually, probably, you don't want to use the triangles. You prefer the lower perimeter. But this guy is interesting.

Anyway, this is a fun space. What we kind of really care about is-- this is called the Pareto curve. It's the, if I give you some bound on perimeter, what's the smallest area I can get? This is the best bound we know so far. And we know the Pareto curves below this thing, but can you go lower? We don't know. So these are some fun examples, but actually, the big questions are still pretty much unsolved here.

Even if you restrict to-- I think it's fair to restrict to wrappings of the sphere where every point on the sphere gets covered by a stretched path. That seems like a nice, natural class to think about. But we have no idea what that hull space looks like. Can you do better than these particular attempts? Looks like we're doing pretty good, but we don't even know how far left you can go. So a pretty neat question. And that is Mozart Kugel. Any questions? Other than, can I have some?

**AUDIENCE:** Where can we get some?

**PROFESSOR:** Where can you get some. That's a good question. Most European airports sell Mirabell. So if you're flying through Europe, it's the place to go. Does anyone know if you can get them locally? I think maybe.

**AUDIENCE:** I think Cardullo's.

**PROFESSOR:** Cardullo's that's a good place. In Harvard Square. They have a lot of imported chocolates. So, yeah, I think they have Mirabell.

**AUDIENCE:** [INAUDIBLE]. There's a lot of imported chocolates and confectioneries in the Pru.

**PROFESSOR:** In the Prudential Center. All right. These I think you have to go to Salzburg. Even in Austria they're hard to get. So these are very special. I'll give Marty my second one. All right. That's all for today.

Hey, there's one thing I forgot to talk about in lecture which I think is really cool so I'm going to add a little bit to what we were talking about. We have all these wrappings of the sphere but I never proved to that they're actually contractive. How do you prove that all those wrappings are contractive?

Well, there's a cool theorem we proved just for this purpose. I'll call it the Cauchy Arm Lemma on a growing sphere. It's a nice little connection to Cauchy's Arm Lemma which we talked about in the context of Cauchy's rigidity theorem for polyhedral. Now, remember, Cauchy's Arm Lemma is about you have some convex chain-- and actually it applies both on the plane and the sphere.

So we're thinking of an open chain but we had the property that even if you add the closing edge it's convex. And then we looked at a flex of the chain where all of the angles increased and we said, if all the angles increase, then this distance increases, the endpoint-- the distance between the endpoints increases. That's regular Cauchy's Arm Lemma.

But now I want to look at not growing the angles, but growing the sphere that the linkage lives on. So think of this in the spherical case, which we'll-- imagine this thing is on a sphere and now I transform it by taking a larger sphere, larger radius, and drawing exactly the same edge lengths and angles. Here, I'll draw it a little bit curved because in reality these edges are great circular arcs.

So this edge length here matches this length, this angle matches this angle, this edge length matches this edge length, and so on. But I draw it on a bigger sphere. This theorem says, that the distance between the endpoints will increase also in that case. So that's kind of neat.

And in the limit, if you take a ginormous sphere, all the way-- if you take a really, really big sphere, it becomes a plane. So if I take something on a sphere and then I actually draw it on the plane with the same angle-- so this planar angle matches this spherical angle and this length matches this spherical length-- then, in particular, this distance will increase. And in the reverse, if I take a planar convex chain and I redraw it on some sphere, the distance will go down provided this thing is also convex. So that's the Lemma.

Now, let me show you how to use that to prove that all those wrappings are contractive. It's really easy once you have this Lemma. Let's start with the source wrapping. So remember, we had a sphere, the north pole, and we took all these shortest paths from the north pole to the south pole. All of those lines-- I want to claim that-- and when you unfold this you get a really big disk and all those lines become straight.

So what I need to show is that between every pair of points, the distance decreases when I go from flat to sphere. Which is exactly what happened over here. I went from flat to the sphere, this distance decreased. So that's how it's going to be useful. If I take two points, like, say, this point and this point. Any two points. They live on one of these spokes that goes to the center. So draw those two segments to the center which was the north pole here. So if I take the corresponding two points up here, I'm looking at that path and that path.

Well, that's a convex chain and the distance between them-- between those two points that I started with, say, x and y, is exactly the closing distance of that convex chain. So here's a convex chain, it matches the angles and the lengths of this convex chain and the plane. Therefore, this distance is larger than this distance. And so this is a contractive map because it works for any two points x and y.

So that's the source wrapping. With slightly more effort, slightly more interesting example, is the petal wrappings. So think about one petal. So let's say I just take-- in fact, just think about half of a petal. So we had a petal look something like that going to the south pole here. And so we bisected this thing and then we took lots of ribs like that. So that's one half petal.

So when you unfold, it looks like this. Take any two points on the half petal. I'll show, first of all, that the half petal is contractive. So I take two points, say, this one and this one, call them x and y. Now I take this rib, this rib and this rib. Those are the stretched paths again. They map to whatever up here. Again, it's a convex chain and so this distance increases when you go to the sphere because it's convex in both scenarios.

So that's half petals. Now, when you worry about a full petal and you want to show the whole wrapping is expansive, you actually use a different method. So let's say we take a whole petal here. And I want to show not only any pair of points within a single half petal contract but, in fact, any two vertices, any two points contract. So I'm going to take x and some other point. Now I'll call it z.

If I did the-- just looked at the stretched paths, that's no longer convex. I can't use the same trick. But here's a new trick. I take a straight line, a shortest path from x to z, which gives me a new point y in the middle. Now I know that this distance contracts because that's within a single half petal. So by the previous argument, and by looking at that little chain, I know by this Arm Lemma that distance contracts, I know that this distance contracts.

So what about the whole distance? Well, if I look at the distance from x to z-- that's what I care about-- let's say, in the plane, this equals the sum of those distances. Because we drew a straight line, it equals the distance from x to y plus the distance from y to z. Now, I mapped these onto the sphere, call that d prime is the spherical distance. I know that when I look at d prime I contract. So d prime of xy is going to be smaller for xy, it's going to be smaller for yz.

What I really care about is the distance of the sphere between x and z. But this is the triangle inequality. If you have a triangle xyz, the distance from x to z is always, at most, the distance from x to y plus the distance from y to z. That holds in any metric space. So, in particular, it holds on the sphere. So this is sort of trivial. This is triangle inequality.

And this was the contractiveness that we already prove. And this is just equal. And so we get that the new distance is a contraction of the old distance. So that works on a full petal. It actually-- this exact same technique works if you have a full petal unfolding. Maybe you have something like this. And you take-- finish it off-- you take two points anywhere on any two petals, like here and here, you take the shortest path between them and again, you could argue, well, this is going to contract because it is in a single half petal, this is going to contract because it's in a single have petal. Therefore, that sum will also contract by exactly the same argument. And you can use that for the comb wrapping too.

This is kind of neat with this little Lemma about growing spheres and showing the distances increase, you get to prove that all of these wrappings are contractive and therefore all these things I was claiming were working really do work. You can formalize that in a simple little way.