6.896
18.1 4-21-04
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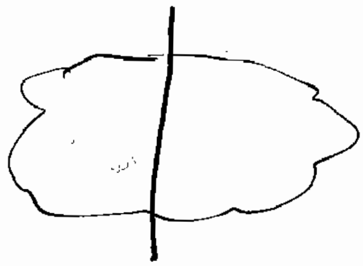
# DIVIDE-AND-CONQUER LAYOUT

~~A GENERAL ALGORITHM FOR LAYOUT. PRODUCES LOW AREA LAYOUTS FOR ANY BOUNDED DEGREE TREE OR PLANAR GRAPH, + NEAR-OPTIMAL FOR ALL~~

A GENERAL LAYOUT ALGORITHM,
NEAR OPTIMAL LAYOUT FOR ALL GRAPHS
OPTIMAL FOR BOUNDED DEGREE TREES + PLANAR GRAPHS

~~SEPARATORS~~
~~IDEA:~~
IDEA:



LAYOUT(G):

(1) FIND SMALL NUMBER OF EDGES THAT DISCONNECT THE GRAPH INTO NEARLY EQUAL PIECES. (CAN BE TOUGH)

(2) RECURSIVELY LAYOUT THE TWO HALVES

(3) PUT THE TWO HALVES TOGETHER, + WIRE IT UP.

SEPARATORS

~~DEFN: G has an~~

DEFN: $G$ a graph $G=(V,E)$

$S: \mathbb{Z} \to \mathbb{Z}$

$G$ has an $S$-separator (is $S$-separable) if

a) $G$ has 1 vertex, or

b) ~~let $q = |V|$~~

$\exists$ a set $A \subseteq E$ s.t. $|A| < S(|V|)$

and $(V, E-A)$ is two disconnected graphs

~~$G_1, G_2$~~ $G_1 = (V_1, E_1)$

$G_2 = (V_2, E_2)$

s.t. $|V_1| \geq |V|/3$ and $|V_2| \geq |V|/3$

[neither $G_1$ nor $G_2$ has > twice the nodes of the other]

and $G_1 + G_2$ are $S$-separable.

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~~Example~~:

Thm: Binary trees are 1-separable.

proof: pick a root
travel down the tree looking for a node that is the ancestor of $\frac{n}{3}$ to $\frac{2n}{3}$ nodes.

~~that node's parent edge disconnects the tree~~

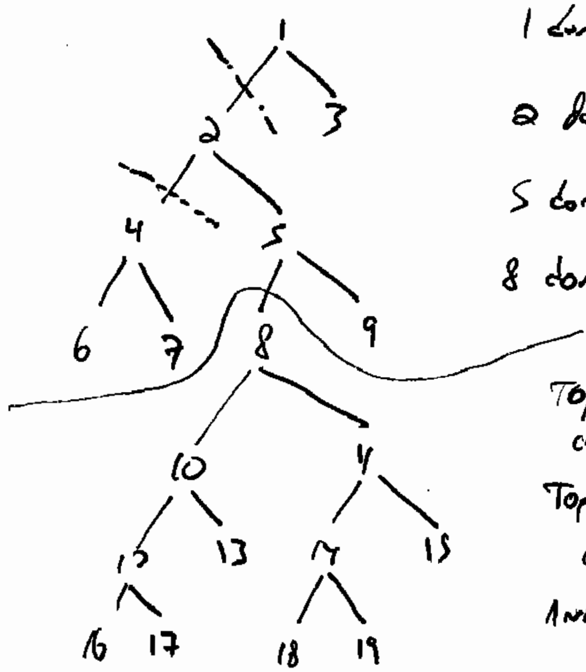
case A: this subtree is between $\frac{n}{3}$ + $\frac{2n}{3}$ nodes
done

case B: ~~subtree too small. we d~~
Subtree too big. $> \frac{2n}{3}$ nodes

One of two children is at least half ~~$\frac{2n}{3}$~~ the nodes,
so go that path

case C: too small. we don't go there.

Example..




1 dominates 19 nodes - too big

2 dominates 17 nodes too big

5 dominates 13 nodes, too big

8 dominates 12 nodes - ok

Top half is 8 nodes
cut here ...... into 3 + 5

Top 1/4 is 5 nodes
cut here —·—·—·—

And so forth

SAVE THIS TREE

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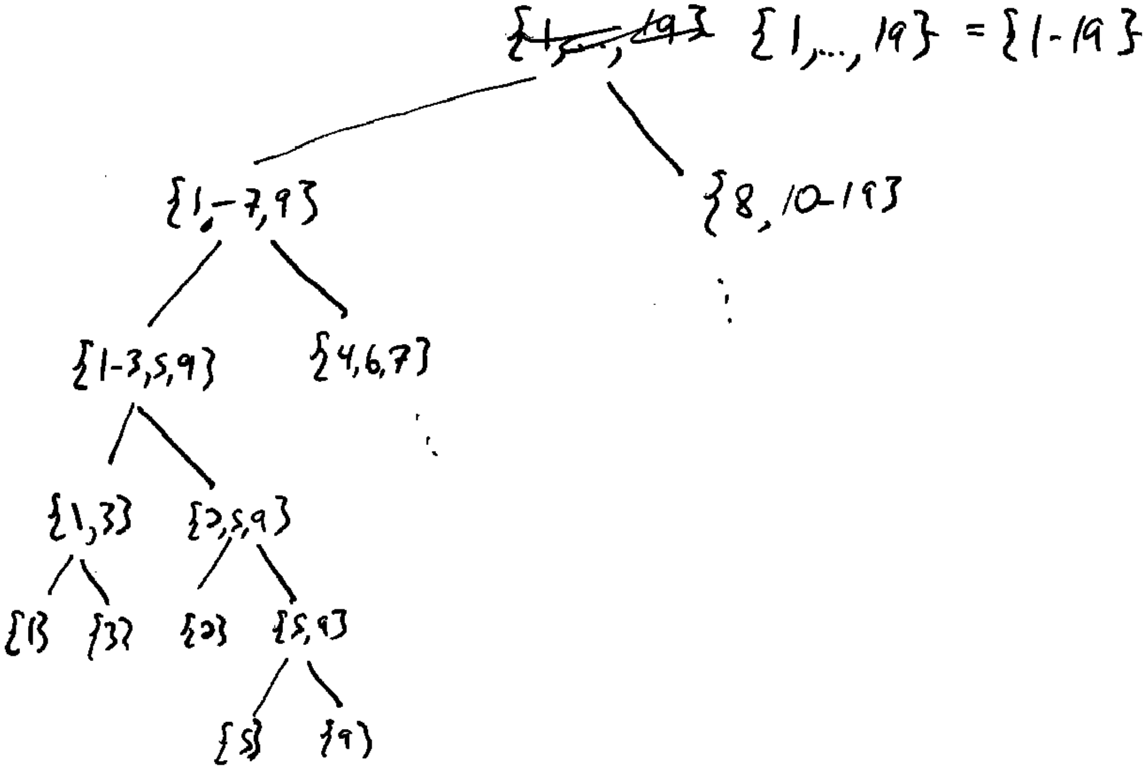
Def'n:
A partition tree of a GRAPH is a ~~tree~~ ~~where each~~
Defined by example:




[its a tree even if G is not a tree]

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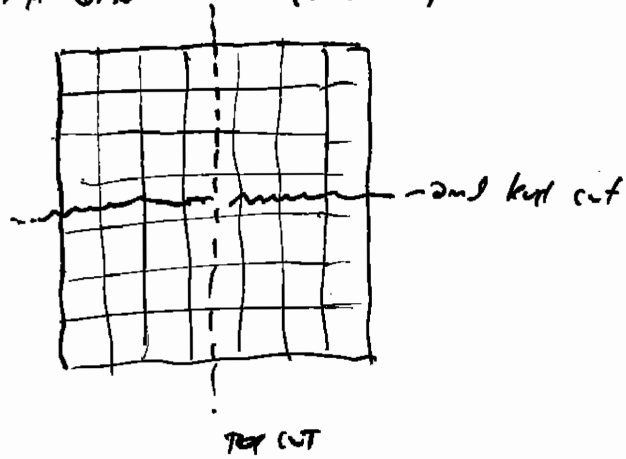
Example: A GMD is $O(\sqrt{n})$ separable




Defn: G has a strong S-separator if the sizes of the subgraphs are at most $\frac{|V|+1}{2}$.

[cut exactly in half even, otherwise as close as possible]

Example: ~~a good~~ a grid of size $2^i \times 2^i$ is strongly $\sqrt{n}$ separable

Claim: not surprising. We'll show ~~$n^\epsilon$ separable $\Rightarrow$~~ $S$ is $\Omega(n^\epsilon)$ implies $S$-separable $\Rightarrow$ strongly separable

Defn: $\Gamma$ (gamma) defined as

$$\Gamma_S(n) = S(n) + S\left(\tfrac{2}{3}n\right) + S\left(\tfrac{4}{9}n\right) + \cdots$$

$$= \sum_{i=0}^{\lceil \log_{3/2} n \rceil} S\left(\left(\tfrac{2}{3}\right)^i n\right)$$

Example: $S(n) = n^\alpha$ then

$$\Gamma_S(n) = n^\alpha + \left(\tfrac{2}{3}n\right)^\alpha + \left(\tfrac{4}{9}n\right)^\alpha + \cdots = n^\alpha \cdot \left(1 + \tfrac{2^\alpha}{3} + \tfrac{4^\alpha}{9} + \cdots\right)$$

$$= n^\alpha \cdot \frac{1}{1-\left(\frac{2}{3}\right)^\alpha} = O(n^\alpha)$$

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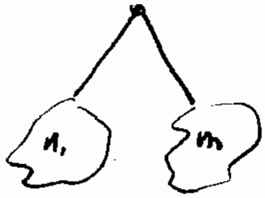
proof by induction: pick $t<|V|$

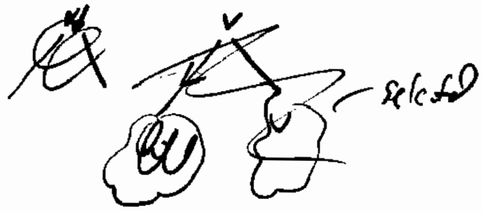



if $n_1 \le t$ then add the left subtree to our selected set + go right to pick $t-n_1$ elts.

~~if $n_1+n_2 \ge t$ then pick one side~~
if $n_1 > t$ then go left
don't use the right subtree; pick $n_1$ elts from left.

claim: ~~the edges connecting our selected sets to anything else~~
~~[illegible]~~ $\le \Gamma_S(n)$ edges connecting selected sets to anything else.

pf.




~~If u is selected then at most $S(n)$ edges connect u to anything else~~ non-selected set

~~If u is not selected then at most $S(n)$ edges connect u to the other vertices to the selected set~~.

~~If v is~~ .


If u is selected then at most $S(n)$ of the edges to u are needed to connect u to non-selected nodes.

If u is not selected, then at most $S(n)$ of those edges are needed to connect u to the selected set.

The total number is then no more than $S(n)+S(\frac{2}{3}n)+S(\frac{4}{9}n)$

$= \Gamma_S(n)$.

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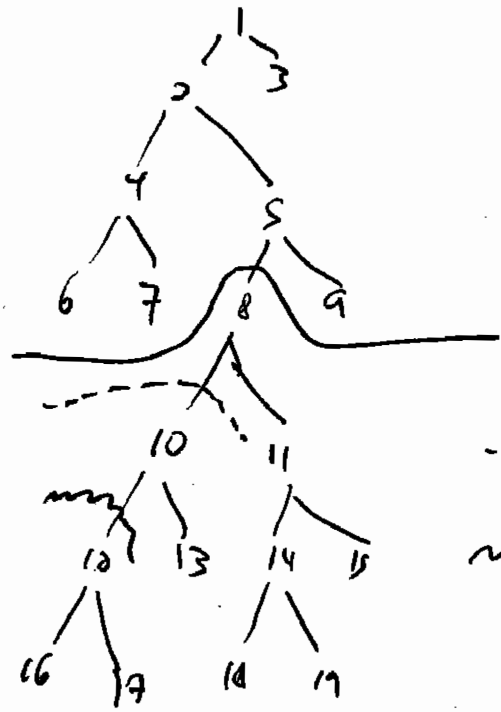
Example: Binary trees are 1-separable ⇒ they are ~~less~~ strongly $\log_{3/2}$-separable.



~~Break into 10 + 9 nodes,~~

Select 10 nodes

select top half + need to select 2 nodes

----- is too big, select neither

~~~ is just right

~~3 nodes are cut~~

3 edges cut if best 10 nodes

$\Gamma_1(19) = \lceil \log_{3/2} 20 \rceil = 8$ so we did ok.
~~~

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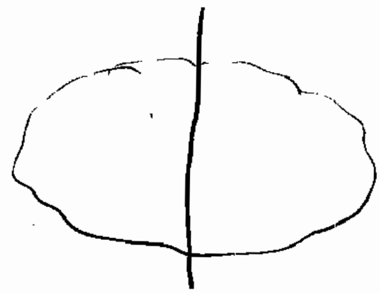
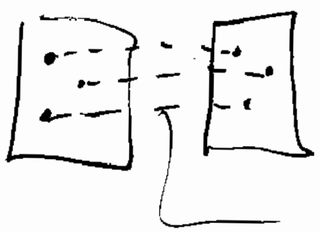
Back to layout algorithm
1) Separate
2) recurse
3) reassemble

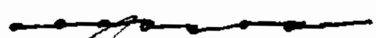


cut in 1/2 with separator
did recursive layout of halves
now what?

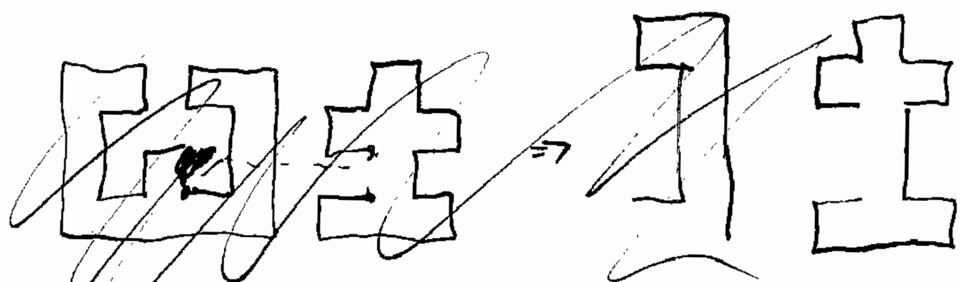


need to connect these edges.

e.g. ~~here is a tree~~



~~May have this recursive layout~~



~~insertChannels~~ stretch layout vertically to make space
leave a space between for vertical routing

Example: $S(n) = O(1)$ then

$$\Gamma_S(n) = O(\log n)$$

Example: $S(n) = \lg n$ then

$$\Gamma_{\lg} n = \lg n + \lg \tfrac{2}{3} n + \lg \tfrac{4}{9} n + \cdots$$
$$= \lg n + \Gamma_{\lg}(\tfrac{2}{3} n)$$
$$= \lg^2 n$$

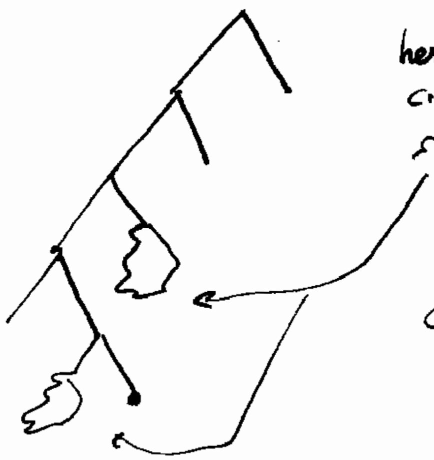
Lemma: if $G$ is $S$-separable then $G$ is strongly $\Gamma_S$ separable.

~~proof: for any $t \le |V|$ $\exists$ a path in the partition tree~~

Proof. Build a partition tree for $G$ achieving $S$-separation

For any $t < |V|$ we will find ~~a collection of nodes in the partition tree~~ a path from the root of the partition tree to a leaf, & some subset of the siblings add up to exactly $t$ nodes

e.g.



here is the P.T. and we can take any collection of the siblings. pick these two to get exactly $t$

call these the selected sets

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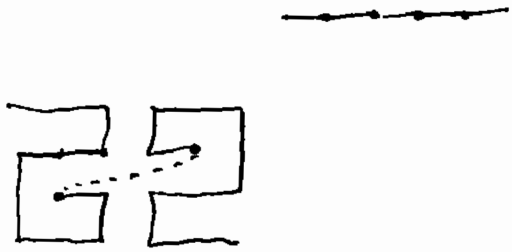
Example: layout of a linear array.

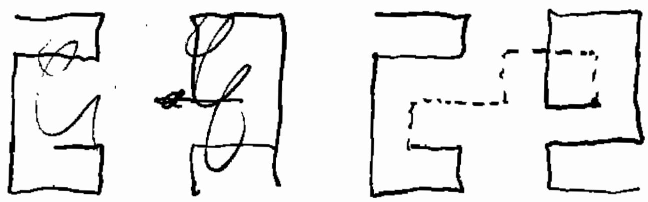


↖ two recursive layouts

---- need to connect

stretch vertically to bring ---- to edge

stretch horizontally between the holes



(n a power of 4)

Def'n $\Delta_S(n) = S(n) + 2S(n/4) + 4S(n/16) \ldots$

$= S(n) + 2\Delta_S(n/4)$

example:

~~S(n) = n^α then~~

[State as facts w/o proof]

$\Delta_{n^\alpha}$

$S(n) = n^\alpha$ $\qquad \Delta_S(n) =$

$\Delta_S(n) = n^\alpha + 2\left(\frac{n}{4}\right)^\alpha + 4\left(\frac{n}{16}\right)^\alpha = O(\sqrt{n})$

$\Delta_{n^\alpha}(n) = O(\sqrt{n})$ if $\alpha < \frac{1}{2}$

$\Delta_{n^\alpha}(n) = O(n^\alpha)$ if $\alpha > \frac{1}{2}$

$\Delta_{n^{1/2}}(n) = O(\sqrt{n}\lg n)$

If $\alpha < \frac{1}{2}$ then the terms grow faster than the bottom

$=$ $\qquad$ so $\Delta_S(n) = O(\sqrt{n})$

If $\alpha > 1/2$ then $\qquad \Delta_S(n) = O(n^\alpha)$

If $\alpha = \frac{1}{2}$ then $\qquad \Delta_S(n) = \Delta_r(n) = O(\sqrt{n}\lg n)$

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Thm: $S(n)$ monotonically non decreasing

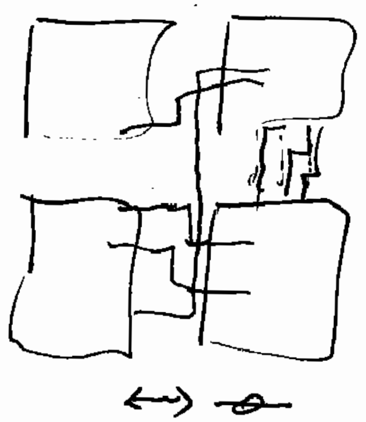
A graph of $n$ nodes with a strong ~~$S(n)$ separator~~ $S$-separator can be laid out in a square with side length $O(\max(\sqrt{n}, \Delta_S(n)))$

Induction on $n$, assume $n$ a power of 4

Claim side length of $\sqrt{n} + 6\Delta_S(n)$ good enough

base case: easy 1 node

induction: divide in half & half again




$$W(n) = 2\left(\frac{\sqrt{n}}{2} + 6\Delta_S(n/4)\right) + S(n)$$

but $\Delta_S(n) = S(n) + 2\Delta_S(n/4)$

$$= \sqrt{n} + 6\Delta_S(n/4) \qquad \square$$

$$H(n) = S(n) + S(n/2) + 2H(n/4) = O(S(n)) + H(n/4)$$

Analysis: $S(n) = O(n^\alpha)$ for $\alpha < \frac{1}{2}$: $H(n) = O(\sqrt{n}) + H(n/4) = O(\sqrt{n} + \sqrt{\frac{n}{4}} + \sqrt{\ }$