Prof. P.A. Parrilo Spring 2006

MIT 6.972
Algebraic methods and semidefinite programming
Homework assigment # 1

Date Given: February 20th, 2006
Date Due: March 2nd, 4PM

P1. [***20 pts***] Classify the following statements as true of false. A proof or counterexample is required.

Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping, and $K \subset \mathbb{R}^n$ a cone.

(a) If K is convex, then $\mathcal{A}(K)$ is convex.
(b) If K is solid, then $\mathcal{A}(K)$ is solid.
(c) If K is pointed, then $\mathcal{A}(K)$ is pointed.
(d) If K is closed, then $\mathcal{A}(K)$ is closed.

Do the answers change if $\mathcal{A}$ is injective and/or surjective? How?

P2. [***20 pts***] Consider the following SDP:

$$\min x \qquad \text{s.t.} \quad \begin{bmatrix} x & 1 \\ 1 & y \end{bmatrix} \succeq 0.$$

(a) Draw the feasible set. Is it convex?
(b) Write the dual SDP.
(c) Is the primal strictly feasible? Is the dual strictly feasible?
(d) What can you say about strong duality? Are the results consistent with Theorem 3.1 in the Vandenberghe & Boyd paper?

P3. [***20 pts***] Given a set $\mathcal{S} \subseteq \mathbb{R}^n$ that strictly contains the origin, we define the *dual set* $\mathcal{S}^o \subseteq \mathbb{R}^n$ as:

$$\mathcal{S}^o = \{y \in \mathbb{R}^n \,|\, y^T x \leq 1, \quad \forall x \in \mathcal{S}\}.$$

(a) Let $\mathcal{S}$ be the feasible set of an SDP, i.e.,

$$\mathcal{S} = \{x \in \mathbb{R}^m \,|\, \sum_{i=1}^m x_i A_i \preceq A_0\},$$

where $A_0 \succ 0$. Find a convenient description of $\mathcal{S}^o$. Can you optimize a linear function over $\mathcal{S}^o$?

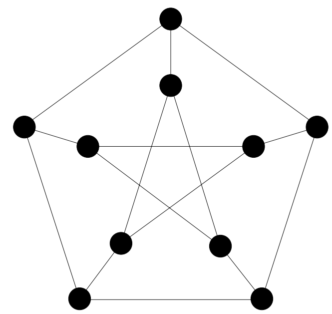

Figure 1: Petersen graph

P4. [***20 pts***] Consider the graph given in Figure 1, known as the Petersen graph.

- **(a)** Compute the SDP upper bound on the size of its largest stable subset (i.e., the Lovász theta function).
- **(b)** Compute the SDP upper bound on its maximum cut.
- **(c)** Are these bounds tight? Can you find the true optimal solutions?

We suggest to use a suitable parser (e.g., YALMIP) for the formulation in MATLAB of the corresponding SDP.

P5. [***20 pts***] Consider the primal-dual pair of relaxations for optimization problems presented in the notes for Lecture 3, Section 3.

- **(a)** Verify that they indeed constitute a primal-dual pair of SDPs.
- **(b)** Why does the solution of (9) provide a lower bound on the objective?
- **(c)** What is the relationship between the matrix X and the variable of the original optimization problem?