Bayes Networks

6.872/HST.950

What Probabilistic Models Should We Use?

- Full joint distribution
  - Completely expressive
  - Hugely data-hungry
  - Exponential computational complexity
- Naive Bayes (full conditional independence)
  - Relatively concise
    - Need data ~ (#hypotheses) × (#features) × (#feature-vals)
    - Fast ~ (#features)
  - Cannot express dependencies among features or among hypotheses
  - Cannot consider possibility of multiple hypotheses co-occurring

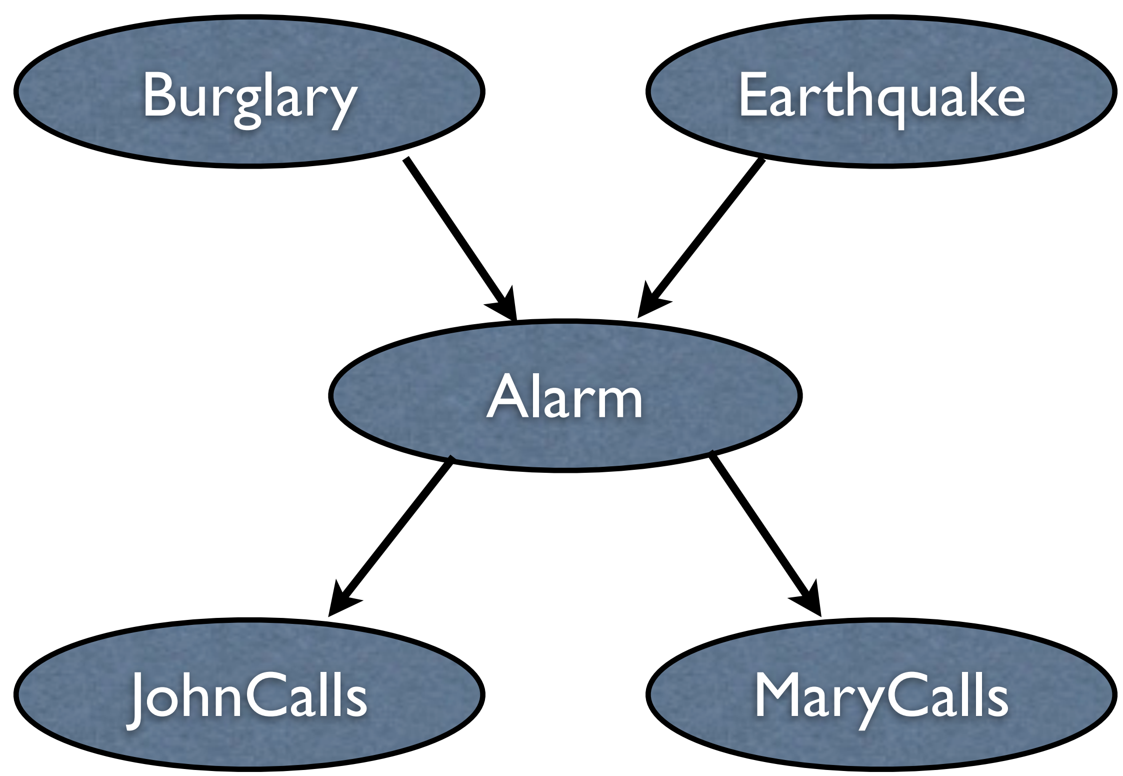
Bayesian Networks

(aka Belief Networks)

- Graphical representation of dependencies among a set of random variables
  - Nodes: variables
  - Directed links to a node from its *parents*: direct probabilistic dependencies
  - Each X_i has a conditional probability distribution, $P(X_i|\text{Parents}(X_i))$, showing the effects of the parents on the node.
  - The graph is directed (DAG); hence, no cycles.
- This is a language that can express dependencies between Naive Bayes and the full joint distribution, more concisely
  - Given some new evidence, how does this affect the probability of some other node(s)? P(X|E) —*belief propagation/updating*
  - Given some evidence, what are the most likely values of other variables? $\operatorname{argmax}_{\boldsymbol{X}} P(\boldsymbol{X}|\boldsymbol{E})$ —*MAP explanation*

Burglary Network

(due to J. Pearl)

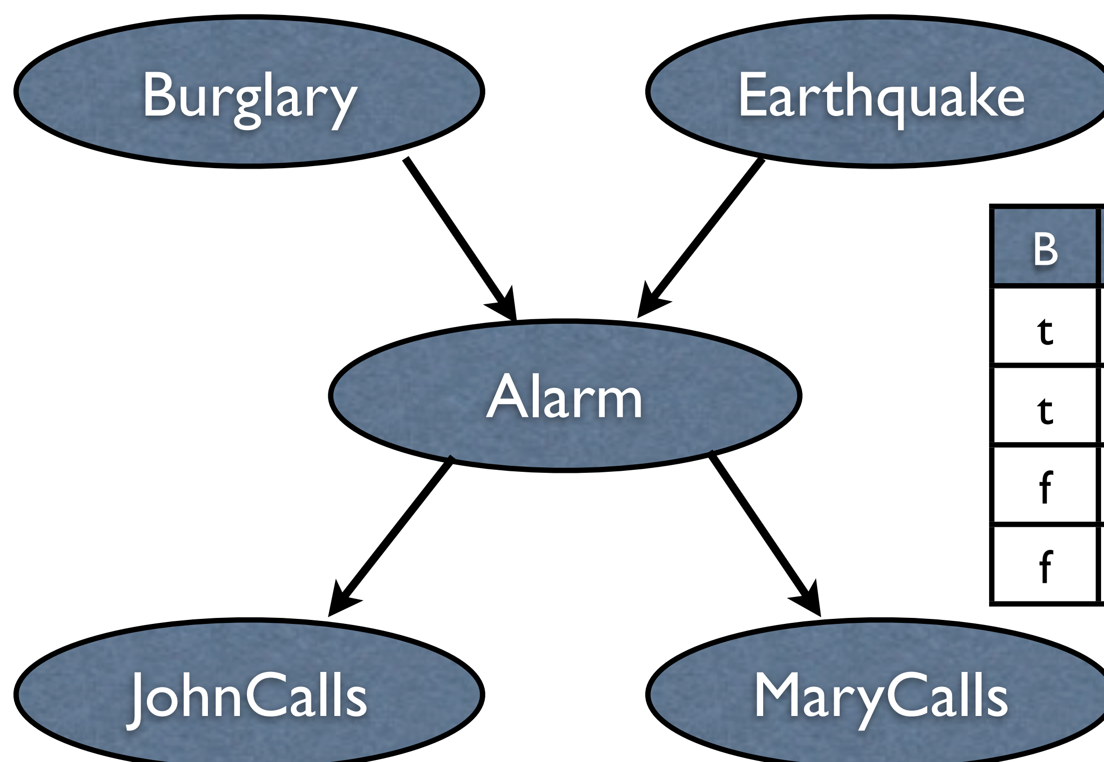


Burglary Network

(due to J. Pearl)

P(B)
0.001

P(E)
0.002

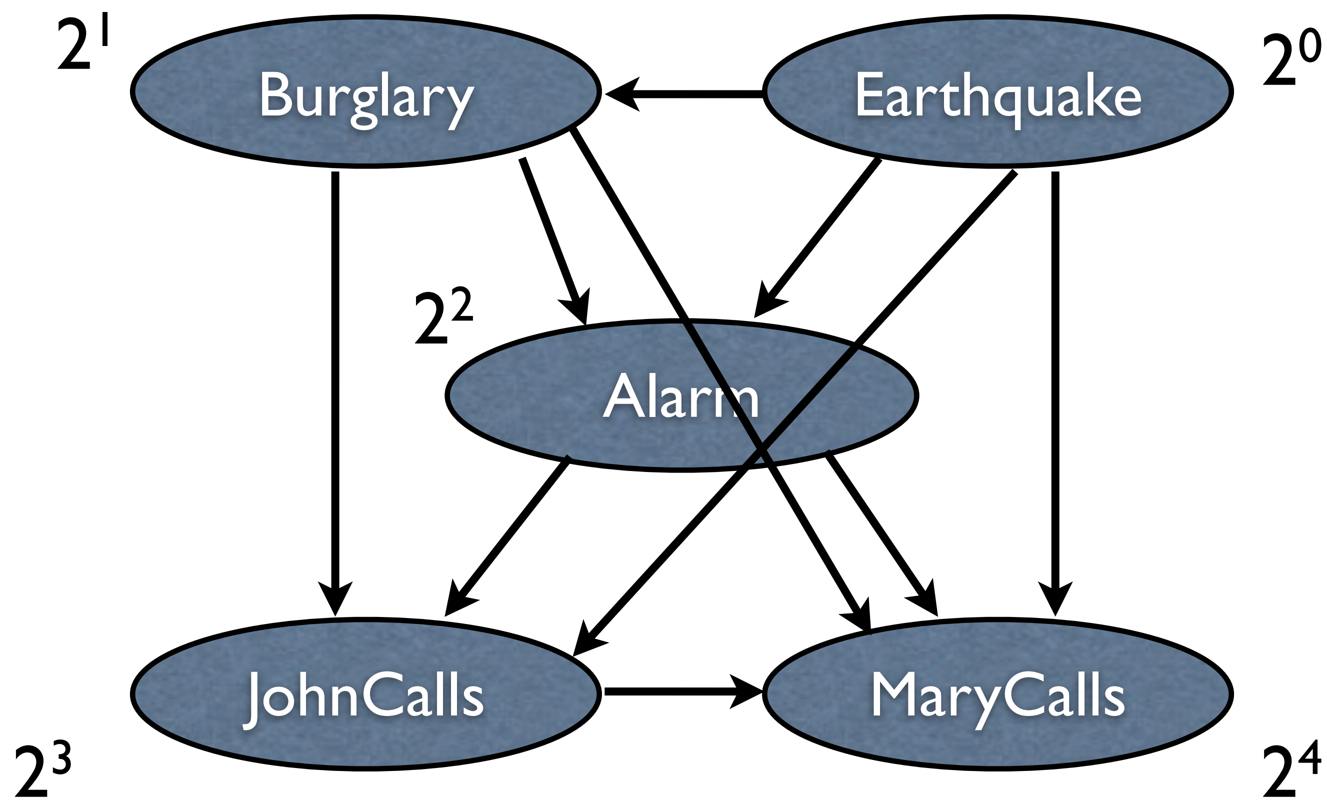


B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

t	0.90
f	0.05

t	0.70
f	0.01

If everything depends on everything



- This model requires just as many parameters as the full joint distribution!

Computing the Joint Distribution from a Bayes Network

- As usual, we abuse notation:

  $P(X_1 = x_1 \wedge \ldots \wedge X_n = x_n)$ is written as $P(x_1, \ldots, x_n)$

- $P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i|\mathrm{Par}(X_i))$

- E.g., what's the probability that an alarm has sounded, there was neither an earthquake nor a burglary, and both John and Mary called?

$$
\begin{aligned}
& P(j \wedge m \wedge a \wedge \neg b \wedge \neg e) \\
&\quad = \quad P(J|a)P(m|a)P(a|\neg b \wedge \neg e)P(\neg e)P(\neg b) \\
&\quad = \quad 0.90 \times 0.70 \times 0.001 \times 0.999 \times 0.998 = 0.00062
\end{aligned}
$$

Requirements for Constructing a BN

- Recall that the definition of the conditional probability was

  $P(x|y) = P(x \wedge y)/P(y)$

- and thus we get the *chain rule*,

  $P(x \wedge y) = P(x|y)P(y)$

- Generalizing to *n* variables,

  $P(x_1, \ldots, x_n) = P(x_n|x_{n-1}, \ldots, x_1)P(x_{n-1}, \ldots, x_1)$

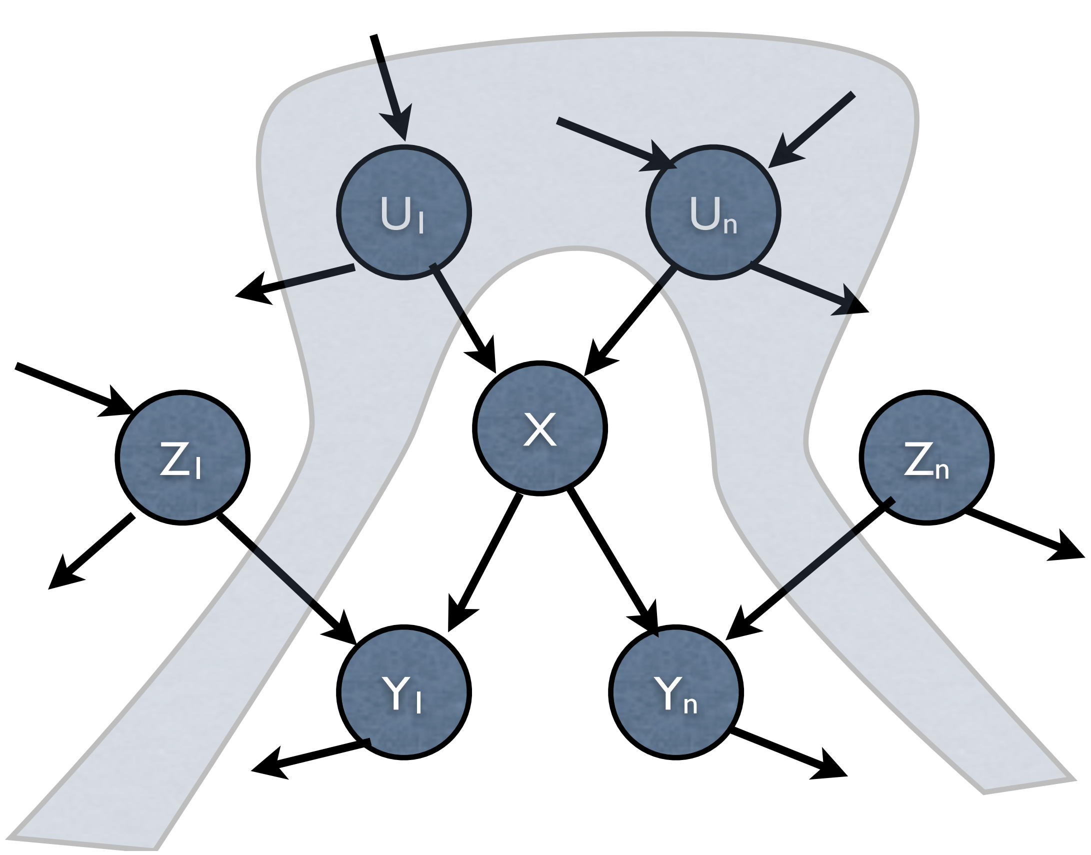
- and repeatedly applying this idea,

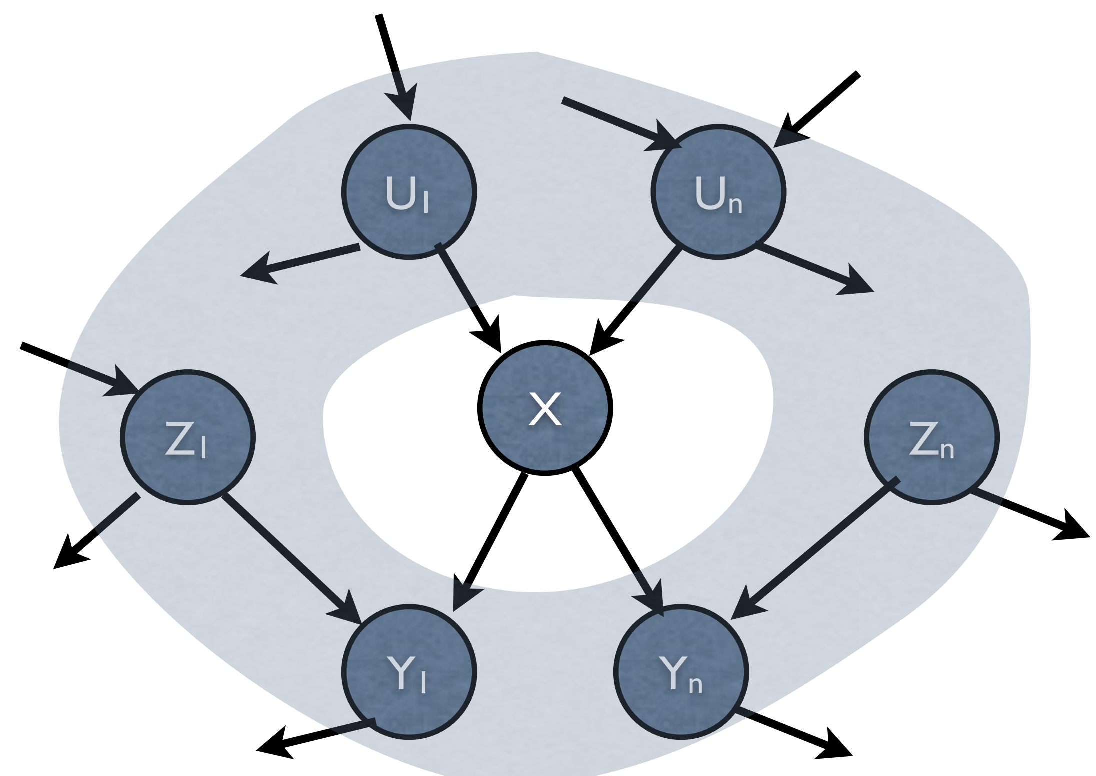
$$
\begin{aligned}
P(x_1, \ldots, x_n) &= P(x_n|x_{n-1}, \ldots, x_1)P(x_{n-1}|x_{n-2}, \ldots, x_1) \cdots P(x_2|x_1)P(x_1) \\
&= \prod_{i=1}^{n} P(x_i|x_{i-1}, \ldots, x_1) \\
&= \prod_{i=1}^{n} P(x_i|\mathrm{Par}(x_i))
\end{aligned}
$$

- This "works" just in case we can define a partial order so that

  $\mathrm{Par}(X_i) \subseteq \{X_{i-1}, \ldots, X_1\}$

Topological Interpretations



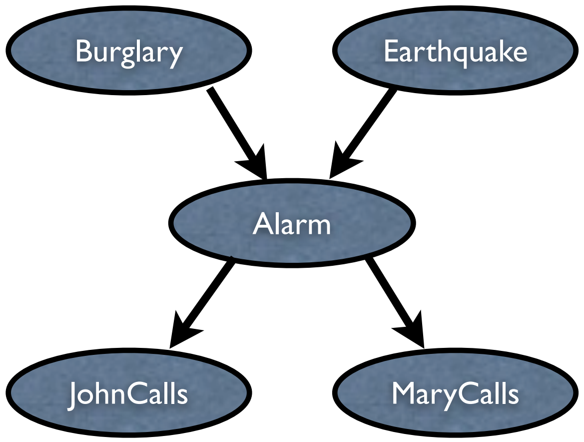


A node, *X*, is *conditionally independent* of its non-descendants, Z_i, given its parents, U_i.

A node, *X*, is *conditionally independent* of all other nodes in the network given its *Markov blanket*: its parents, U_i, children, Y_i, and children's parents, Z_i.

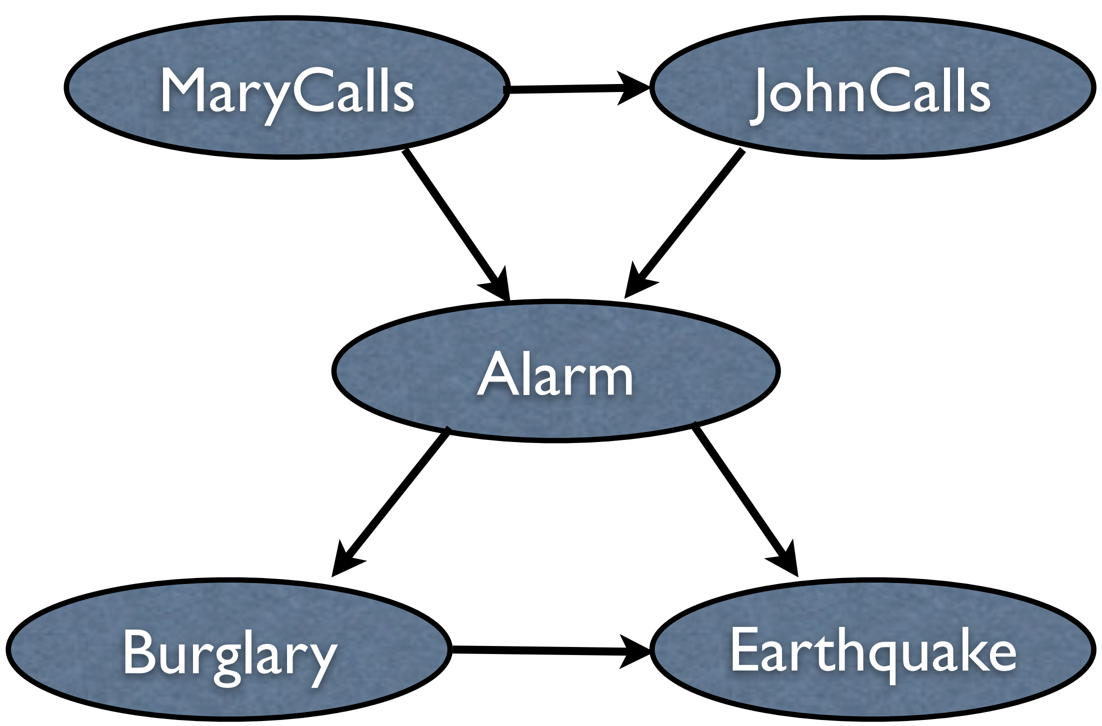
BN's can be Compact

- For a network of 40 binary variables, the full joint distribution has 2^{40} entries (> 1,000,000,000,000)
- If $|Par(x_i)| \leq 5$, however, then the 40 (conditional) probability tables each have ≤ 32 entries, so the total number of parameters $\leq 1{,}280$
- Largest medical BN I know (Pathfinder) had 109 variables! $2^{109} \approx 10^{36}$

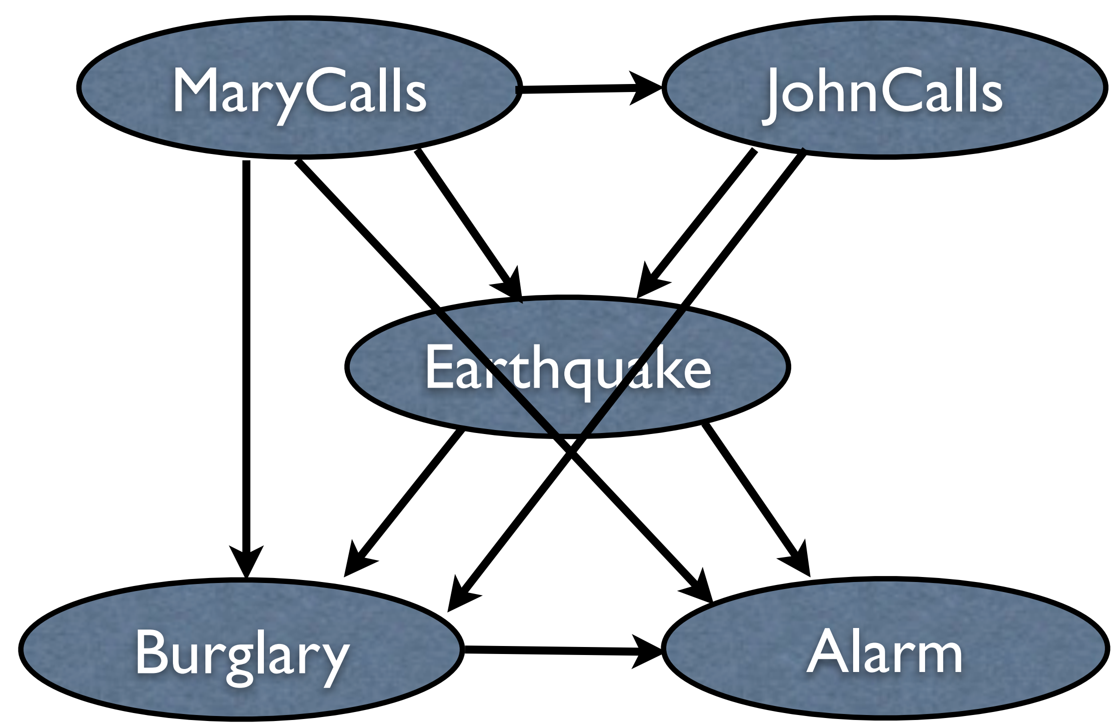


How *Not* to Build BN's

- With the wrong ordering of nodes, the network becomes more complicated, and requires more (and more difficult) conditional probability assessments



Order: M, J, A, B, E



Order: M, J, E, B, A

Simplifying Conditional Probability Tables

- Do we know any structure in the way that Par(x) "cause" x?
- If each destroyer can sink the ship with probability $P(s|d_i)$, what is the probability that the ship will sink if it's attacked by both?

  $(1 - P(s|d_1, d_2)) = (1 - P(s|d_1))(1 - P(s|d_2))\ (1 - l)$
- For |Par(x)| = *n*, this requires O(n) parameters, not $O(k^n)$

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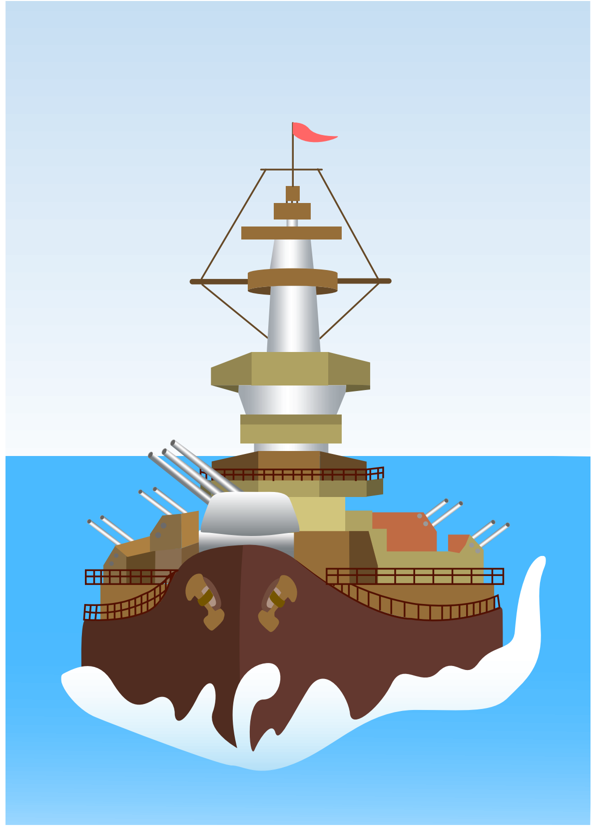

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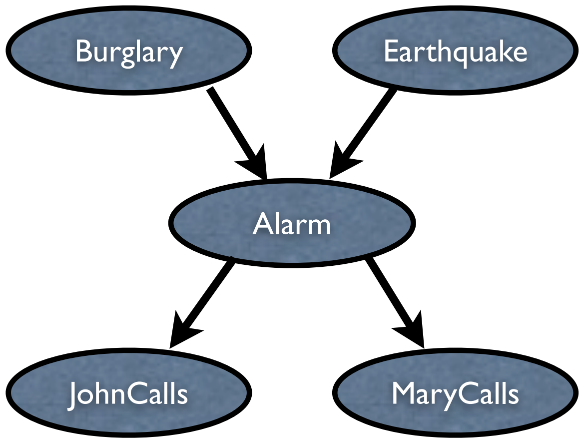
Inference

- Recall the two basic inference problems: Belief propagation & MAP explanation
  - Trivially, we can enumerate all "matching" rows of the joint probability distribution
  - For *poly-trees* (not even <u>undirected</u> loops—i.e., only one connection between any pair of nodes; like our Burglary example), there are efficient linear algorithms, similar to constraint propagation
  - For arbitrary BN's, all inference is NP-hard!
    - Exact solutions
    - Approximation



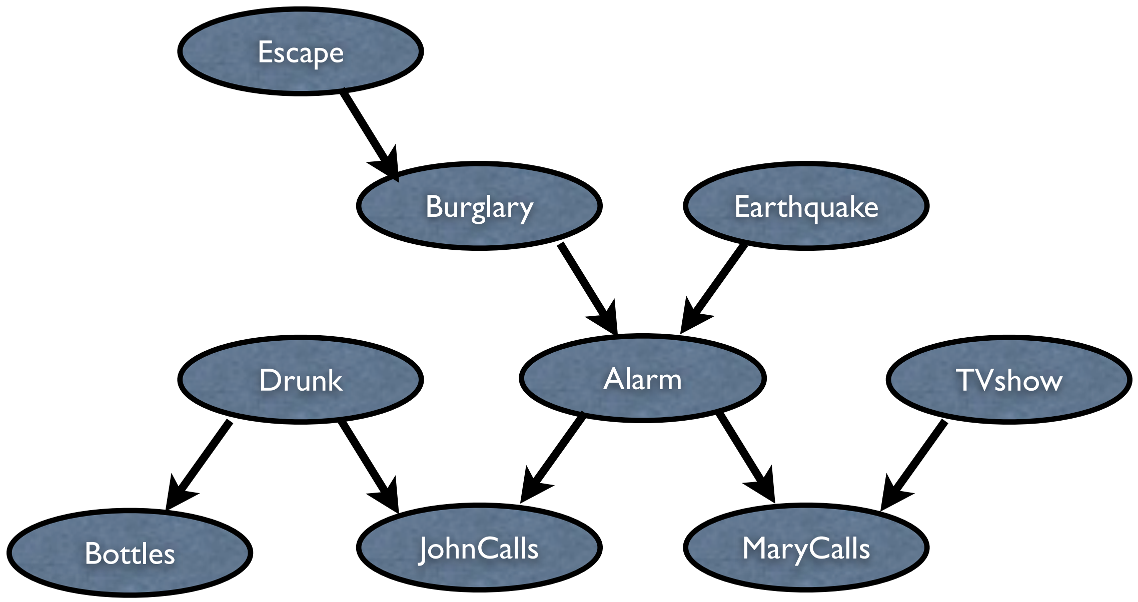
Exact Solution of BN's
(Burglary example)

$$
\begin{aligned}
P(b|j,m) &= \alpha \sum_e \sum_a P(b)P(e)P(a|b,e)P(j|a)P(m|a) \\
&= \alpha P(b) \sum_e P(e) \sum_a P(a|b,e)P(j|a)P(m|a) \\
P(\boldsymbol{B}|j,m) &= \alpha\{0.00059224, 0.0014919\} \approx \{\, 0.28, 0.72\}
\end{aligned}
$$

- Notes:
  - Sum over all “don’t care” variables
  - Factor common terms out of summation
  - Calculation becomes a sum of products of sums of products ...

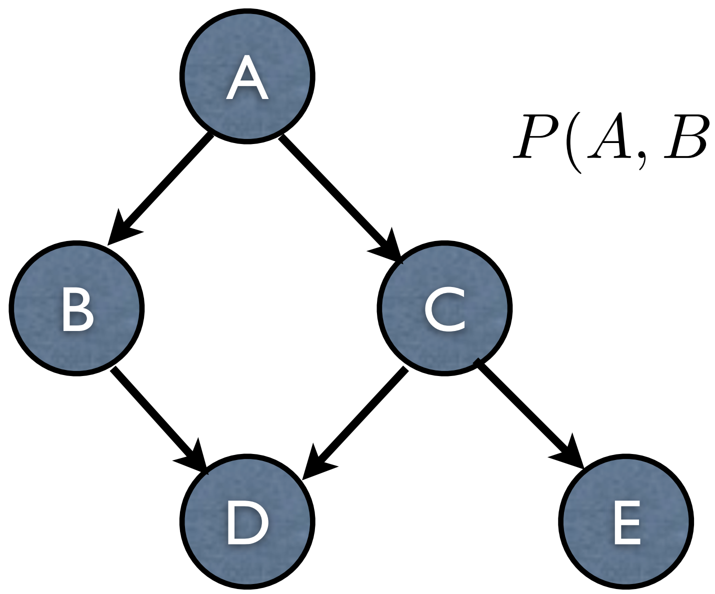
Poly-trees are easy

- Singly-connected structures allow propagation of observations via single paths
- “Down” is just use of conditional probability
- “Up” is just Bayes rule
- Formulated as message propagation rules
- Linear time (network diameter)
- Fails on general networks!



Exact Solution of BN's

(non-poly-trees)



$$P(A,B,C,D,E) = P(A)P(B|A)P(C|A)P(D|B,C)P(E|C)$$

- What is the probability of a specific state, say A=t, B=f, C=t, D=t, E=f?

  $$p(a,\neg b,c,d,\neg e) = p(a),p(\neg b|a)p(c,a)p(d|\neg b,c)p(\neg e|c)$$

- What is the probability that E=t given B=t?

  $$p(e|b) = p(e,b)/p(b)$$

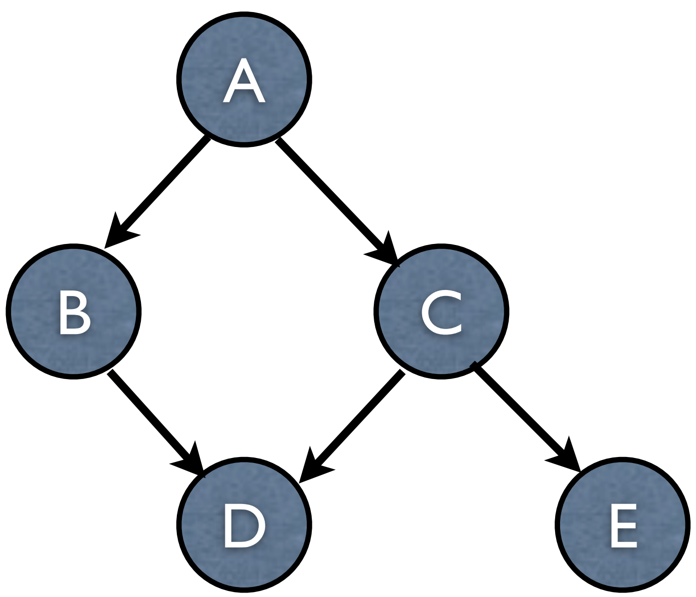
- Consider the term *P(e,b)*

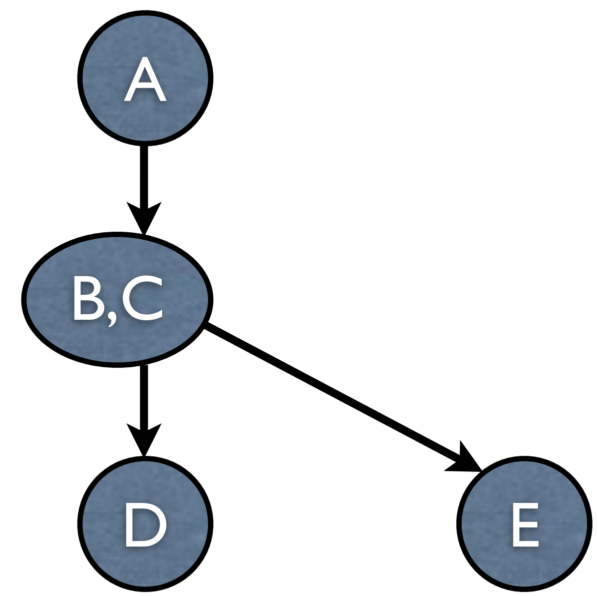
$$\begin{aligned} P(e,b) &= \sum_{A,C,D} P(A,b,C,D,e) \\ &= \sum_{A,C,D} P(A)P(b|A)P(C|A)P(D|b,C)P(e|C) \\ &= \sum_{C} P(e|C)\left(\sum_{A} P(A)P(C|A)P(b|A)\right)\left(\sum_{D} P(D|b,C)\right) \end{aligned}$$

Alas, optimal factoring is NP-hard

- 12 instead of 32 multiplications (even in this small example)

Other Exact Methods





- *Join-tree*: Merge variables into (small!) sets of variables to make graph into a poly-tree. Most commonly-used; aka *Clustering, Junction-tree, Potential*)
- *Cutset-conditioning*: Instantiate a (small!) set of variables, then solve each residual problem, and add solutions weighted by probabilities of the instantiated variables having those values
- ...
- All these methods are essentially equivalent; with some time-space tradeoffs.

Approximate Inference in BN's

- Direct Sampling—samples joint distribution
- Rejection Sampling—computes P($\mathbf{X}$|e), uses *ancestor* evidence nodes in sampling
- Likelihood Weighting—like Rejection Sampling, but weights by probability of *descendant* evidence nodes
- Markov chain Monte Carlo
    - Gibbs and other similar sampling methods

Direct Sampling

function Prior-Sample(bn) returns an event sampled from bn
  inputs: bn, a Bayes net specifying the joint distribution $\mathbf{P}(X_1, \ldots X_n)$
 x := an event with *n* elements
 for i = 1 to *n* do
  x_i := a random sample from $P(X_i|Par(X_i))$
 return x

$$\lim_{n\to\infty} \frac{N_{PS}(x_1,\ldots,x_n)}{N} = P(x_1,\ldots,x_n) \qquad P(x_1,\ldots,x_m) \approx \frac{N_{PS}(x_1,\ldots,x_m)}{N}$$

- From a large number of samples, we can estimate all joint probabilities
  - The probability of an event is the fraction of all complete events generated by PS that match the partially specified event
    - hence we can compute all conditionals, etc.

Rejection Sampling

```
function Rejection-Sample(X, e, bn, N) returns an estimate of P(X|e)
  inputs: bn, a Bayes net
          X, the query variable
          e, evidence specified as an event
          N, the number of samples to be generated
  local: K, a vector of counts over values of X, initially 0

 for j = 1 to N do
  y := PriorSample(bn)
  if y is consistent with e then
   K[v] := K[v]+1 where v is the value of X in y
 return Normalize(K[X])
```

- Uses PriorSample to estimate the proportion of times each value of X appears in samples that are consistent with e
- But, most samples may be irrelevant to a specific query, so this is quite inefficient

Likelihood Weighting

- In trying to compute P(X|**e**), where **e** is the *evidence* (variables with known, observed values),
  - Sample only the variables other than those in **e**
  - Weight each sample by how well it predicts **e**

$$
\begin{aligned}
S_{\mathrm{WS}}(\boldsymbol{z}, \boldsymbol{e}) w(\boldsymbol{z}, \boldsymbol{e}) &= \prod_{i=1}^{l} P(z_i | \mathrm{Par}(Z_i)) \prod_{i=1}^{m} P(e_i | \mathrm{Par}(E_i)) \\
&= P(\boldsymbol{z}, \boldsymbol{e})
\end{aligned}
$$

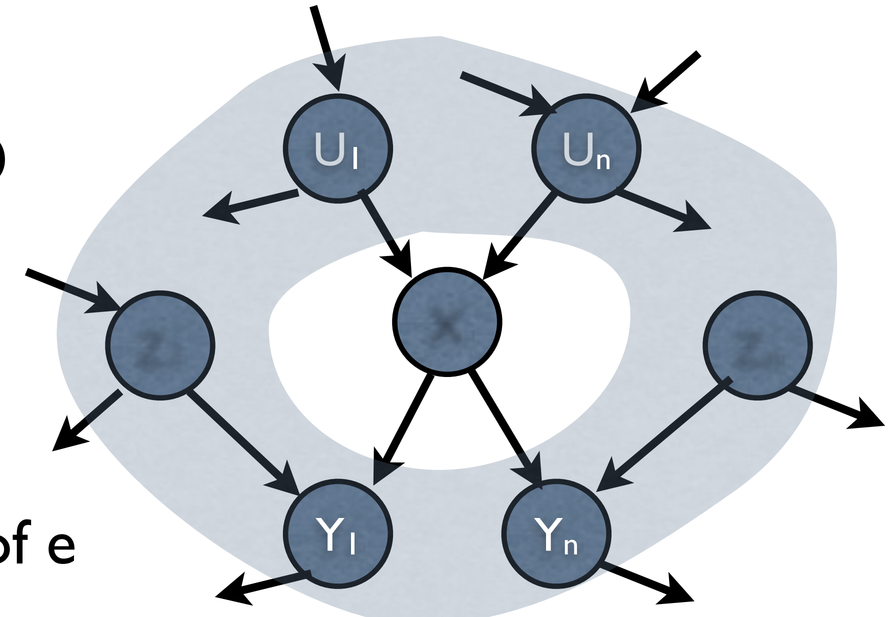
Likelihood Weighting

$$
\begin{aligned}
S_{\mathrm{WS}}(\boldsymbol{z},\boldsymbol{e})w(\boldsymbol{z},\boldsymbol{e}) &= \prod_{i=1}^{l} P(z_i|\mathrm{Par}(Z_i)) \prod_{i=1}^{m} P(e_i|\mathrm{Par}(E_i)) \\
&= P(\boldsymbol{z},\boldsymbol{e})
\end{aligned}
$$

function Likelihood-Weighting(X, e, bn, N) returns an estimate of P(X|e)
  inputs: bn, a Bayes net
    X, the query variable
    e, evidence specified as an event
    N, the number of samples to be generated
  local: W, a vector of weighted counts over values of X, initially 0
 for j = 1 to N do
  y,w := WeightedSample(bn,e)
  if **y** is consistent with e then
   W[v] := W[v]+w where v is the value of X in **y**
 return Normalize(W[X])

function Weighted-Sample(bn,e) returns an event and a weight
 x := an event with *n* elements; w := 1
 for i = 1 to *n* do
  if X_i has a value x_i in e
   then w := w * P(X_i = x_i | Par(X_i))
   else x_i := a random sample from P(X_i | Par(X_i))
 return x,w

Markov chain Monte Carlo




```
function MCMC(X, e, bn, N) returns an estimate of P(X|e)
    local: K[X], a vector of counts over values of X, initially 0
           Z, the non-evidence variables in bn (includes X)
           x, the current state of the network, initially a copy of e
  initialize x with random values for the vars in Z
  for j = 1 to N do
    for each Zi in Z do
        sample the value of Zi in x from P(Zi|mb(Zi)), given the values of mb(Zi) in x
        K[v] := K[v]+1 where v is the value of X in x
  return Normalize(K[X])
```

- Wander incrementally from the last state sampled, instead of re-generating a completely new sample
- For every unobserved variable, choose a new value according to its probability given the values of vars in it Markov blanket (remember, it's independent of all other vars)
- After each change, tally the sample for its value of X; this will only change sometimes
- Problem: "narrow passages"

Most Probable Explanation

- So far, we have been solving for P(**X**|e), which yields a distribution over all possible values of the x's
- What it we want the *best explanation* of a set of evidence, i.e., the highest-probability set of values for the x's, given e?
- Just *maximize* over the "don't care" variables rather than summing
- This is not necessarily the same as just choosing the value of each x with the highest probability

Rules and Probabilities

- Many have wanted to put a probability on assertions and on rules, and compute with likelihoods
- E.g., Mycin's *certainty factor* framework
  - A (p=.3) & B (p=.7) ==p=.8==> C (p=?)
- Problems:
  - How to combine uncertainties of preconditions and of rule
  - How to combine evidence from multiple rules
- Theorem: There is NO such algebra that works when rules are considered independently.
- Need BN for a consistent model of probabilistic inference

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