Countable Additivity

1 The Principle of Countable Additivity

(Finite) Additivity $p(A \text{ or } B) = p(A) + p(B)$

whenever A and B are incompatible propositions

Countable Additivity $p(A_1 \text{ or } A_2 \text{ or } \ldots) = p(A_1) + p(A_2) + \ldots$

whenever $A_1, A_2, \ldots$ are countably many propositions with A_i and A_j incompatible for $i \neq j$.

2 Against Countable Additivity

- God has selected a positive integer, and that you have no idea which.
- For n a positive integer, what credence should you assign to the proposition, G_n, that God selected n?

Countable Additivity entails that your credences *should remain undefined* (unless you're prepared to give different answers for different choices of n).

Proof: suppose otherwise. Then $p(G_n) = r$, for $r \in [0, 1]$.

- Suppose $r = 0$. By Countable Additivity:

$$\begin{aligned} p(G_1 \text{ or } G_2 \text{ or } G_3 \text{ or } \ldots) &= p(G_1) + p(G_2) + p(G_3) + \ldots \\ &= \underbrace{0 + 0 + 0 + \ldots}_{\text{once for each integer}} \\ &= 0 \end{aligned}$$

- Suppose $r > 0$. By Countable Additivity:

$$\begin{aligned} p(G_1 \text{ or } G_2 \text{ or } G_3 \text{ or } \ldots) &= p(G_1) + p(G_2) + p(G_3) + \ldots \\ &= \underbrace{r + r + r + \ldots}_{\text{once for each integer}} \\ &= \infty \end{aligned}$$

Moral: Countable Additivity entails that there is no way of distributing probability uniformly across a countably infinite set of (mutually exclusive and jointly exhaustive) propositions.

2.1 Infinitesimals to the rescue?

What if we had an infinitesimal value ι with the following property?

$$\underbrace{\iota + \iota + \iota + \ldots}_{\text{once for each positive integer}} = 1$$

Then:

$$\begin{aligned} p(G_1 \text{ or } G_3 \text{ or } G_5 \text{ or } \ldots) &= p(G_1) + p(G_3) + p(G_5) + \ldots \\ &= \underbrace{\iota + \iota + \iota + \ldots}_{\text{once for each positive integer}} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} p(G_2 \text{ or } G_4 \text{ or } G_6 \text{ or } \ldots) &= p(G_2) + p(G_4) + p(G_6) + \ldots \\ &= \underbrace{\iota + \iota + \iota + \ldots}_{\text{once for each positive integer}} \\ &= 1 \end{aligned}$$

So, by (finite) Additivity:

$$p(G_1 \text{ or } G_2 \text{ or } G_3 \text{ or } \ldots) = 2 \text{ (!)}$$

3 For Countable Additivity

- $X, Y \subseteq \mathbb{Z}^+$
- $p(X)$ is the probability that God selects a number in X.
- $p(X|Y)$ is the probability that God selects a number in X given that She selects a number in Y.

Here is a natural way of characterizing $p(X)$ and $p(X|Y)$:

$$\begin{aligned} p(X|Y) &=_{df} \lim_{n \to \infty} \frac{|X \cap Y \cap \{1, 2, \ldots, n\}|}{|Y \cap \{1, 2, \ldots, n\}|} \\ p(X) &=_{df} p(X|\mathbb{Z}^+) \end{aligned}$$

- $p(X)$ is finitely additive but not countably additive.
- $p(X)$ is not well-defined for arbitrary sets of integers.[1]

Also, there is a set S and a partition E_i of $\mathbb{Z}^+$ such that:

- $p(S) = 0$
- $p(S|E_i) \geq 1/2$ for each E_i.

  Example:

  $S = \{1^2, 2^2, 3^2, \dots\}$; E_i be the set of powers of i (whenever i which is not a power of any other positive integer). In other words:

$$
\begin{array}{ccc}
S & = & \{1, 4, 9, 16, 25, \dots\} \\
E_1 & = & \{1\} \\
E_2 & = & \{2, 4, 8, 16, 32, \dots\} \\
E_3 & = & \{3, 9, 27, 81, 243, \dots\} \\
\text{[No } E_4\text{, since } 4 = 2^2\text{]} & & \\
E_5 & = & \{5, 25, 125, 625, 3125, \dots\} \\
 & \vdots &
\end{array}
$$

3.1 Is this really so bad?

Yes. There is a sequence of bets $B_{E_1}, B_{E_2}, B_{E_3}, B_{E_5}, \dots$ such that:

- you are confident that you ought to take each of the bets,
- you are 100% confident that you will lose money if she takes them all.

$\boldsymbol{B_{E_i}}$ **:** Suppose God selects a number in E_i. Then you'll receive \$2 if the selected number is in S and you'll be forced to pay \$1 if the selected number is not in S. (If the selected number is not in E_i, then the bet is called off and no money exchanges hands.)

Problems of this general form are inescapable: *they will occur whenever a probability function on a countable set of possibilities fails to be countably additive.*

[1]For instance, when X consists of the integers k such that $2^m \leq k < 2^{m+1}$, for some even m.

4 The Two-Envelope Paradox

- Two envelopes:
  - one contains $\$n$, for n chosen at random from $\mathbb{Z}^+$.
  - the other contains $2n$.
- You are handed one of the envelopes, but don't know which.
- Then you are offered the chance to switch. Should you switch?

An argument for switching:

Say that your envelope contains $\$k$. If k is odd, you should switch. If k is even, there's a 0.5 chance that the other envelope has $\$2k$ and a 0.5 chance that the other envelope has $\$k/2$. So:

$$\begin{aligned} EV(\text{switch}) &= \$k/2 \cdot 0.5 + \$2k \cdot 0.5 = 5/4 \cdot \$k \\ EV(\text{not switch}) &= \$k \end{aligned}$$

5 Broome's Variant of the Paradox

- Two envelopes:
  - Toss a die until it lands One or Two. If the die first lands One or Two on the kth toss, place 2^{k-1} in the first envelope.
  - Place twice that amount in the other envelope.