The Ordinals

1 How We'll Get to the Ordinals

Ordering $\rightarrow$ Total Ordering $\rightarrow$ Well-Ordering $\rightarrow$ Well-Order Type $\rightarrow$ Ordinal

2 Orderings

Think of $x < y$ as meaning "x precedes y". We say that $<$ is an **ordering** on set A if and only if for any $a, b, c \in A$:

Asymmetry If $a < b$, then not-$(b < a)$.

Transitivity If $a < b$ and $b < c$, then $a < c$.

3 Total Orderings

A **total ordering** $<$ on A is an ordering on A such that for any distinct elements a, b of A:

Totality $a < b$ or $b < a$

4 Well-Orderings

A **well-ordering** $<$ of A is a total ordering on A such that:

Well-Ordering Every non-empty subset S of A has a $<$-smallest member.

5 Well-order types

The orderings $<_1$ and $<_2$ are of the same type if they are isomorphic.*

*Let $<_1$ be an ordering on A and $<_2$ be an ordering on B. Then $<_1$ is **isomorphic** to $<_2$ if and only if there is a bijection f from A to B such that, for every x and y in A, $x <_1 y$ if and only if $f(x) <_2 f(y)$.

6 The Ordinals

ordinal	name of ordinal	well-order type represented
$\{\}$	0	
$\{0\}$	$0'$	$\lvert$
$\{0, 0'\}$	$0''$	$\lvert\lvert$
$\{0, 0', 0''\}$	$0'''$	$\lvert\lvert\lvert$
$\vdots$	$\vdots$	
$\{0, 0', 0'', 0''', \ldots\}$	ω	$\lvert\lvert\lvert\lvert\lvert \ldots$
$\{0, 0', 0'', 0''', \ldots, \omega\}$	ω'	$\lvert\lvert\lvert\lvert\lvert \ldots \lvert$
$\{0, 0', 0'', 0''', \ldots, \omega, \omega'\}$	ω''	$\lvert\lvert\lvert\lvert\lvert \ldots \lvert\lvert$
$\vdots$	$\vdots$	$\vdots$

7 Constructing the Ordinals

Construction Principle At each stage, we introduce a new ordinal, namely: the set of all ordinals that have been introduced at previous stages.

Open-Endedness Principle However many stages have occurred, there is always a "next" stage, that is, a first stage after every stage considered so far.[†]

[†]It is important to interpret the Open-Endedness Principle as entailing that there is no such thing as "all" stages—and therefore deliver the result that there is no such thing as "all" ordinals.