The Ordinals (Part II)

1 The First Few Ordinals

ordinal	name of ordinal	well-order type represented
$\{\}$	0	
$\{0\}$	$0'$	$\vert$
$\{0, 0'\}$	$0''$	$\vert\vert$
$\{0, 0', 0''\}$	$0'''$	$\vert\vert\vert$
$\vdots$	$\vdots$	
$\{0, 0', 0'', 0''', \ldots\}$	ω	$\vert\vert\vert \ldots$
$\{0, 0', 0'', 0''', \ldots, \omega\}$	ω'	$\vert\vert\vert \ldots \vert$
$\{0, 0', 0'', 0''', \ldots, \omega, \omega'\}$	ω''	$\vert\vert\vert \ldots \vert\vert$
$\vdots$	$\vdots$	$\vdots$

2 Constructing the Ordinals

Construction Principle At each stage, we introduce a new ordinal, namely: the set of all ordinals that have been introduced at previous stages.

Open-Endedness Principle However many stages have occurred, there is always a "next" stage, that is, a first stage after every stage considered so far.[1]

3 Ordering the Ordinals

The ordinals are well-ordered by the following precedence relation:

$$\alpha <_o \beta \leftrightarrow_{df} \alpha \in \beta$$

[1]It is important to interpret the Open-Endedness Principle as entailing that there is no such thing as "all" stages—and therefore deliver the result that there is no such thing as "all" ordinals.

4 Representing Well-Order Types

Since every ordinal is a set of ordinals, the elements of an ordinal are always well-ordered by $<_o$. So we may set forth the following:

Representation Principle Each ordinal represents the well-order type that it itself instantiates under $<_o$.

5 Some Definitions

- $\alpha' = \alpha \cup \{\alpha\}$
- A **successor ordinal** is an ordinal α such that $\alpha = \beta'$ for some β.
- A **limit ordinal** is an ordinal that is not a successor ordinal.

6 Ordinal Addition

The intuitive idea: A well-ordering of type $(\alpha + \beta)$ is the result of starting with a well-ordering of type α and appending a well-ordering of type β at the end.

Formally:

$$\begin{array}{lllll} \alpha & + & 0 & = & \alpha \\ \alpha & + & \beta' & = & (\alpha+\beta)' \\ \alpha & + & \lambda & = & \bigcup\{\alpha+\beta : \beta < \lambda\} \text{ } (\lambda \text{ a limit ordinal}) \end{array}$$

- Ordinal addition is associative: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
- Ordinal addition is *not* commutative: it is not generally the case that $\alpha + \beta = \beta + \alpha$.

7 Ordinal Multiplication

The intuitive idea: A well-ordering of type $(\alpha \times \beta)$ is the result of starting with a well-ordering of type β and replacing each position in the ordering with a well-ordering of type α.

Formally:

$$\begin{array}{rcrcl} \alpha & \times & 0 & = & 0 \\ \alpha & \times & \beta' & = & (\alpha \times \beta) + \alpha \\ \alpha & \times & \lambda & = & \bigcup\{\alpha \times \beta : \beta < \lambda\} \text{ (λ a limit ordinal)} \end{array}$$

- Ordinal multiplication is associative: $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$.
- Ordinal multiplication is *not* commutative: it is not generally the case that $\alpha \times \beta = \beta \times \alpha$.

8 Some Additional Operations

- Exponentiation:

$$\begin{array}{rcl} \alpha^0 & = & 0' \\ \alpha^{\beta'} & = & (\alpha^\beta) \times \alpha \\ \alpha^\lambda & = & \bigcup\{\alpha^\beta : \beta < \lambda\} \text{ (λ a limit ordinal)} \end{array}$$

- Tetration:

$$\begin{array}{rcl} {}^0\alpha & = & 0' \\ {}^{\beta'}\alpha & = & ({}^\beta\alpha)^\alpha \\ {}^\lambda\alpha & = & \bigcup\{{}^\beta\alpha : \beta < \lambda\} \text{ (λ a limit ordinal)} \end{array}$$

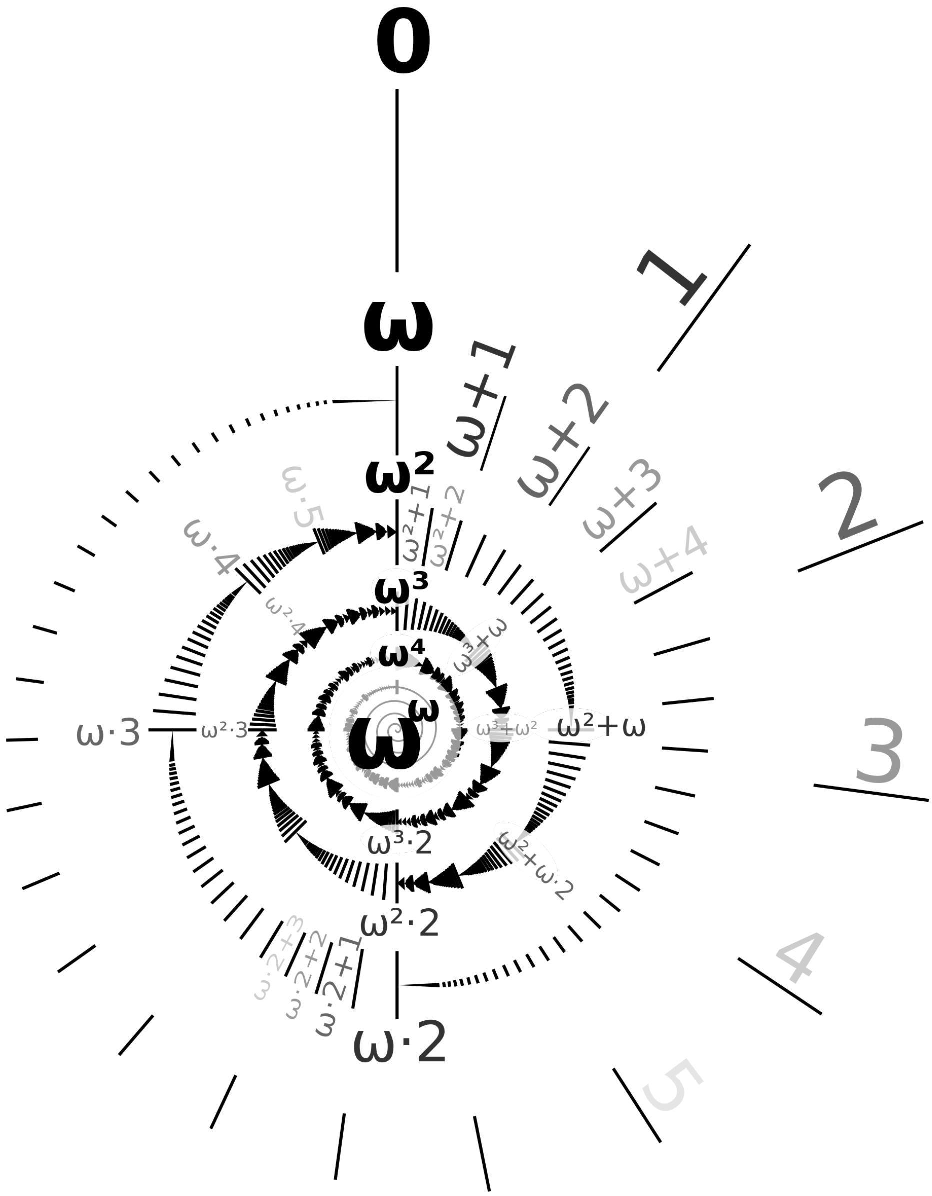
- And so forth…

9 Some Additional Ordinals

ordinal	members	well-order type represented
ω	$\{0, 0', \dots\}$	$\vert\vert\vert\dots$
$\omega + 0'$	$\{0, 0', \dots, \omega\}$	$\vert\vert\vert\dots\vert$
$\omega + \omega$	$\{0, 0', \dots, \omega, \omega + 0', \dots\}$	$\vert\vert\vert\dots\vert\vert\vert\dots$
$\omega + \omega + \omega$	$\{0, 0', \dots, \omega, \omega + 0', \dots, \omega + \omega, \omega + \omega + 0', \dots\}$	$\vert\vert\vert\dots\vert\vert\vert\dots\vert\vert\vert\dots$
$\omega \times \omega$ $= \omega^{0''}$	$\{0, \dots, \omega, \dots, \omega + \omega, \dots, \omega + \omega + \omega, \dots\ \dots\}$	$\underbrace{\vert\vert\vert\dots\vert\vert\vert\dots\vert\vert\vert\dots\ \dots}_{\omega \text{ times}}$
$(\omega \times \omega) + \omega$	$\{0, \dots, \omega \times \omega, (\omega \times \omega) + 0', (\omega \times \omega) + 0'', \dots\}$	$\underbrace{\vert\vert\vert\dots\vert\vert\vert\dots\vert\vert\vert\dots\ \dots}_{\omega \text{ times}}\vert\vert\vert\dots$
$(\omega \times \omega) + \omega + \omega$	$\{0, \dots, \omega \times \omega, (\omega \times \omega) + 0', \dots (\omega \times \omega) + \omega, (\omega \times \omega) + \omega + 0' \dots\}$	$\underbrace{\vert\vert\vert\dots\vert\vert\vert\dots\vert\vert\vert\dots\ \dots}_{\omega \text{ times}}\vert\vert\vert\dots\vert\vert\vert\dots$
$(\omega \times \omega) + (\omega \times \omega)$ $= \ \omega \times \omega \times 0''$	$\{0, \dots, \omega \times \omega, \dots (\omega \times \omega) + \omega, \dots, (\omega \times \omega) + \omega + \omega \dots\ \dots\}$	$\underbrace{\vert\vert\vert\dots\vert\vert\vert\dots\vert\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \underbrace{\vert\vert\vert\dots\vert\vert\vert\dots\vert\vert\vert\dots\ \dots}_{\omega \text{ times}}$
$\omega \times \omega \times 0'''$	$\{0, \dots, \omega \times \omega, \dots (\omega \times \omega) + \omega, \dots, (\omega \times \omega) + \omega + \omega \dots\ \dots$ $(\omega \times \omega) + (\omega \times \omega) \dots (\omega \times \omega) + (\omega \times \omega) + \omega \dots (\omega \times \omega) + (\omega \times \omega) + \omega + \omega \dots\ \dots\}$	$\underbrace{\vert\vert\vert\dots\vert\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \underbrace{\vert\vert\vert\dots\vert\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \underbrace{\vert\vert\vert\dots\vert\vert\vert\dots\ \dots}_{\omega \text{ times}}$
$\omega \times \omega \times \omega$ $= \ \omega^{0'''}$	$\{0 \dots, \omega \times \omega, \dots \omega \times \omega \times 0'', \dots, \omega \times \omega \times 0''', \dots\ \dots\}$	$\underbrace{\underbrace{\vert\vert\dots\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \underbrace{\vert\vert\dots\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \dots}_{\omega \text{ times}}$
ω^ω	$\{0, \dots, \omega, \dots, \omega \times 0'', \dots, \omega \times 0''', \dots\ \dots\}$	[see below]

$$\omega^\omega: \quad \vert\vert\dots\ \underbrace{\vert\vert\dots\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \underbrace{\underbrace{\vert\vert\dots\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \underbrace{\vert\vert\dots\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \dots}_{\omega \text{ times}}\ \underbrace{\underbrace{\underbrace{\vert\vert\dots\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \underbrace{\vert\vert\dots\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \dots}_{\omega \text{ times}}\ \underbrace{\underbrace{\vert\vert\dots\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \underbrace{\vert\vert\dots\vert\vert\dots\ \dots}_{\omega \text{ times}}\ \dots}_{\omega \text{ times}}\ \dots\ \dots}_{\omega \text{ times}}\ \dots$$

10 A Visualization[2]



[2] Source: https://commons.wikimedia.org/wiki/File:Omega-exp-omega-labeled.svg. File made available on Wikimedia under the Creative Commons CC0 1.0 Universal Public Domain Dedication. Pop-up casket (talk); original by User:Fool [CC0].