Omega-sequence Paradoxes

1 What is a Paradox?

A **paradox** is an argument that appears to be valid, and goes from seemingly true premises to a seemingly false conclusion. So we must:

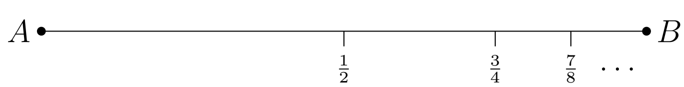
- learn to live with the conclusion;
- learn to live without one of the premises; or
- show that the reasoning is invalid.

An **omega-sequence paradox** is a paradox based on an ω-sequence ($|||\ldots$) or a reverse ω-sequence ($\ldots|||$).

2 Zeno's Paradox[1] [Paradox Grade: 2]

You wish to walk from point A to point B. In order to do so, you must carry out an ω-sequence of tasks:

Task 1:	reach $\frac{1}{2}$ mark
Task 2:	reach $\frac{3}{4}$ mark
Task 3:	reach $\frac{7}{8}$ mark
$\vdots$	$\vdots$
Task n:	reach $\frac{2^n-1}{2^n}$ mark
$\vdots$	$\vdots$



But it's impossible to complete infinitely many tasks in a finite amount of time. So movement is impossible.

[1] This is a variant of one of several paradoxes attributed to ancient philosopher Zeno of Elea, who lived in the 5th Century BC.

3 Thomson's Lamp[2] [Paradox Grade: 3]

You have a lamp with a toggle button: press the button once and the lamp goes on, press it again and the lamp goes off. Here's what happens:

Time to midnight	Status of lamp shortly thereafter
$60s$	off
$30s$	on
$15s$	off
$7.5s$	on
$\vdots$	$\vdots$
$\frac{60}{2^{2n}}s$	off
$\frac{60}{2^{2n+1}}s$	on
$\vdots$	$\vdots$

Is the lamp on or off at midnight?

- For every time the lamp gets turned off before midnight, there is a later time before midnight when it gets turned on. **So the lamp can't be off at midnight**.
- For every time the lamp gets turned on before midnight, there is a later time before midnight when it gets turned off. **So the lamp can't be on at midnight**.

4 The Demon's Game[3] [Paradox Grade: 4]

$P_1, P_2, P_3, \ldots$ take turns answering *aye* or *nay*:

- If exactly n people say *aye* ($n \in \mathbb{N}$), each person gets $\$n$.
- If infinitely many people say *aye*, they all get nothing.

It seems rational for P_k to say *aye*: she can't hurt anyone and might help everyone. But if it's rational for P_k it's rational for everyone. So nobody gets anything.

[2]Thomson's Lamp was devised by the late James Thomson, who was a professor of philosophy at MIT (and was married to the great philosopher Judith Jarvis Thomson).

[3]I learned about this paradox from philosophers Frank Arntzenius, Adam Elga, and John Hawthorne.

5 The Bomber's Paradox[4] [Paradox Grade: 6]

There are infinitely many bombs:

Bomb	When bomb is set to go off
B_0	12:00pm
B_1	11:30am
B_2	11:15am
$\vdots$	$\vdots$
B_k	$\frac{1}{2^k}$ hours after 11:00am
$\vdots$	$\vdots$

Should one of the bombs go off, it will instantaneously disable all other bombs. So a bomb goes off if and only if no bombs have gone off before it:

(0) B_0 goes off $\leftrightarrow$ B_n fails to go off $(n > 0)$.

(1) B_1 goes off $\leftrightarrow$ B_n fails to go off $(n > 1)$.

(2) B_2 goes off $\leftrightarrow$ B_n fails to go off $(n > 2)$.

$\vdots$

($\boldsymbol{k}$) B_k goes off $\leftrightarrow$ B_n fails to go off $(n > k)$.

($\boldsymbol{k+1}$) B_{k+1} goes off $\leftrightarrow$ B_n fails to go off $(n > k + 1)$.

$\vdots$

Will any bombs go off?

[4]This paradox is due to Josh Parsons, who was a fellow at Oxford until shortly before his untimely death in 2017. (It is a version of Bernadete's Paradox.)