Completeness of the SC Rules

We want to show that the system of rules is complete, in the sense that, whenever an argument is valid, the rules make it possible to derive the conclusion from the premisses. We intend to prove the following:

> **Soundness and Completeness Theorem.** A sentence χ is a logical consequence of a set of sentences Γ if and only if there is a derivation by which χ is derived from a premiss set that is included in Γ.

Proof: We've basically already proved the right-to-left direction, which is called the *soundness theorem*, since as we introduced the rules we verified that they ensured that, at each step in a derivation, the sentence we write down in a logical consequence of its premiss set. Our work is to prove the left-to-right direction, which is called the *strong completeness theorem.* To do this, we assume that there isn't any derivation that infers χ from a premiss set that is included in Γ, and we show that there is a NTA under which all the members of Γ are true and χ is false, by showing that there is a complete story that includes Γ and leaves out χ.

We are going to construct a set of sentences Γ_∞ with the following three properties:

(1) $\Gamma \subseteq \Gamma_\infty$.
(2) χ isn't derivable from Γ_∞.
(3) χ is derivable from each set that properly includes Γ_∞.

We'll then verify that any set that satisfies conditions (2) and (3) is a complete story.

The construction of our set Γ_∞ proceeds in stages, analogous to the construction in the compactness theorem. Whereas before we had make sure that, at the end of each stage, the set we had constructed had the property that each finite subset is consistent, now we need to ensure that, at every stage, the set we construct at that stage still has property (2). We begin by enumerating the sentences in an infinite sequence ζ_0, ζ_1, ζ_2, ζ_3,.... We now define a sequence $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \subseteq \ldots$ of sets of sentences, as follows:

$$\Gamma_0 = \Gamma.$$

By hypothesis, Γ_0 has property (2). Given Γ_n with property (2), define:

$$\begin{aligned}\Gamma_{n+1} &= \Gamma_n \cup \{\zeta_n\}, \text{ if that set has property (2)}\\ &= \Gamma_n, \text{ otherwise.}\end{aligned}$$

Let Γ_∞ be the union of the Γ_ns. Γ_∞ has property (1) because it contains Γ_0. If χ were derivable from Γ_∞, it would be derivable from some finite subset of Γ_∞, which means that it would be derivable from Γ_n, for sufficiently large n. Since χ isn't derivable from any of the Γ_ns, it isn't derivable from Γ_∞ either. To verify (3), suppose that $\Delta \supsetneq \Gamma_\infty$, and take ψ

$\in \Delta \sim \Gamma_\infty$. We can find k with $\psi = \zeta_k$. Since $\zeta_k \notin \Gamma_\infty$, $\zeta_k \notin \Gamma_{k+1}$, which can only happen if χ is derivable from $\Gamma_k \cup \{\zeta_k\}$, and hence derivable from Δ.

If follows from (3) that, if θ isn't an element of Γ_∞, then χ must be derivable from $\Gamma_\infty \cup \{\theta\}$. If θ were derivable from Γ_∞, then we could put the two derivations together to get a derivation of χ from Γ_∞, contrary to (2). It follow.s that Γ_∞ is closed under deductive consequence, that is, that every sentence derivable from Γ_∞ is an element of Γ_∞. In particular, every SC theorem is in Γ_∞

If θ isn't in Γ_∞, then, by (3), χ is derivable from $\Gamma_\infty \cup \{\theta\}$, which means, on account of rule CP, that $(\theta \rightarrow \chi)$ is derivable from Γ_∞, and hence an element of Γ_∞.

To confirm that Γ_∞ is a NTA, we have to prove five clauses, one for each connective.

$\neg \varphi$ is in Γ_∞ iff φ isn't in Γ_∞. (⇒) The law of Duns Scotus tells us that $(\neg \varphi \rightarrow (\varphi \rightarrow \chi))$ is in Γ_∞, so that, if Γ_∞ contained both φ and $\neg \varphi$, it would contain χ. **(⇐)** If neither φ nor $\neg \varphi$ were in Γ_∞, $(\varphi \rightarrow \chi)$ and $(\neg \varphi \rightarrow \chi)$ would both be in Γ_∞. But then, since $((\varphi \rightarrow \chi) \rightarrow ((\neg \varphi \rightarrow \chi) \rightarrow \chi))$ is an SC theorem by TH9, χ would be in Γ_∞.

$(\varphi \wedge \psi)$ is in Γ_∞ iff φ and ψ are both in Γ_∞. (⇒) φ and ψ are both derivable from $\{(\varphi \wedge \psi)\}$ by TH13 and TH14, so if $(\varphi \wedge \psi)$ is in Γ_∞, so are φ and ψ. **(⇐)** By TH15, $(\varphi \wedge \psi)$ is derivable from $\{\varphi, \psi\}$, so that, $(\varphi \wedge \psi)$ is in Γ_∞ if φ and ψ are.

$(\varphi \vee \psi)$ is in Γ_∞ iff either or both of φ and ψ are in Γ_∞. (⇒) Suppose that neither φ or ψ is in Γ_∞. Then $(\varphi \rightarrow \chi)$ and $(\psi \rightarrow \chi)$ are both in Γ_∞. By TH12, $((\varphi \vee \psi) \rightarrow \chi)$ is a logical consequence of $\{(\varphi \rightarrow \chi), (\psi \rightarrow \chi)\}$, so that $(\varphi \vee \psi)$ must not be in Γ_∞. **(⇐)** By TH10 and TH11, $(\varphi \vee \psi)$ is derivable both from $\{\varphi\}$ and $\{\psi\}$, so that, if either φ or ψ is in Γ_∞, so is their disjunction.

$(\varphi \rightarrow \psi)$ is in Γ_∞ iff either φ isn't in Γ_∞ or ψ is in Γ_∞. (⇒) If $(\varphi \rightarrow \psi)$ and φ are both in Γ_∞, ψ is in Γ_∞, because Γ_∞ is closed under *modus ponens*. **(⇐)** If φ isn't in Γ_∞, $\neg \varphi$ is in Γ_∞. If $\neg \varphi$ is in Γ_∞, so is $(\varphi \rightarrow \psi)$, on account of TH6. If ψ is in Γ_∞, so is $(\varphi \rightarrow \psi)$, on account of TH2.

$(\varphi \leftrightarrow \psi)$ is in Γ_∞ iff φ and ψ are both in Γ_∞ or neither of them is. (⇒) If $(\varphi \leftrightarrow \psi)$ is in Γ_∞, then, by DC and MP, $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$ is in Γ_∞. It follows, using TH13 and TH14, that $(\varphi \rightarrow \psi)$ and $(\psi \rightarrow \varphi)$ are both in Γ_∞, so that, since Γ_∞ is closed under *modus ponens*, if φ is in Γ_∞ ψ is too, and in ψ is in Γ_∞ φ is too. **(⇐)** If φ and ψ are both in Γ_∞, TH2 assures us that $(\psi \rightarrow \varphi)$ and $(\varphi \rightarrow \psi)$ are both in Γ_∞. It follows by TH15 that $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$ is in Γ_∞, from which it follows by DC that $(\varphi \leftrightarrow \psi)$ is in Γ_∞. If, on the other hand, neither φ nor ψ is in Γ_∞, then both $\neg \varphi$ and $\neg \psi$ are in Γ_∞, and so, by TH6, $(\varphi \rightarrow \psi)$ and $(\psi \rightarrow \varphi)$ are both in Γ_∞. It follows, again by TH15 and DC, that $(\varphi \leftrightarrow \psi)$ is in Γ_∞.

Once we see that Γ_∞ is a complete story, we get our NTA under which all the members of Γ are true and χ is false by setting $\Im(\theta) = 1$ iff $\theta \in \Gamma_\infty$.

The strong completeness theorem immediately entails the so-called *weak completeness theorem*: Every valid SC sentence is an SC theorem.

The theorem provides an alternative proof of the compactness theorem. Suppose that Δ is inconsistent. Then $(P \wedge \neg P)$ is a logical consequence of Δ, which implies, by the strong completeness theorem, that there is a derivation of $(P \wedge \neg P)$ from Δ. The premiss set of the last line of this derivation is a finite subset of Δ; call it Δ^*. Since $(P \wedge \neg P)$ is derivable from Δ^*, it follows by the soundness theorem that $(P \wedge \neg P)$ is a logical consequence. Hence Δ^* is a finite inconsistent subset of Δ.

All in all, this is not a very impressive theorem. It succeeds in its advertised aim, which is to show that the particular system consisting of the five rules PI, CP, MP, MT, and DC, is sound and complete, but it doesn't have a wider significance. It let's us derive the compactness theorem, but we already knew the compactness theorem. It demonstrates that there is a proof procedure for demonstrating the validity of valid SC arguments, but the method of truth tables already gave us a much more powerful result, namely, that there is a decision procedure for testing the validity of SC arguments. There is a genuine benefit to proving the theorem, but it's a benefit that won't emerge until later: Proving the soundness and completeness theorem our deduction system for the sentential calculus will pave the way for the subsequent soundness and completeness theorem for the predicate calculus, where a proof procedure is precisely what one wants.