Question 1

I first prove that if $\ulcorner\mathbf{P} \vee \mathbf{Q}\urcorner \in \Gamma^\star$, then $\mathbf{P} \in \Gamma^\star$ or $\mathbf{Q} \in \Gamma^\star$. I do this by proving the contrapositive — i.e., that if it is not the case that $\mathbf{P} \in \Gamma^\star$ or $\mathbf{Q} \in \Gamma^\star$, then it is not the case that $\ulcorner\mathbf{P} \vee \mathbf{Q}\urcorner \in \Gamma^\star$.

1. Suppose it is not the case that $\mathbf{P} \in \Gamma^\star$ or $\mathbf{Q} \in \Gamma^\star$.
2. Then $\ulcorner\sim \mathbf{P}\urcorner \in \Gamma^\star$ and $\ulcorner\sim \mathbf{Q}\urcorner \in \Gamma^\star$ (by **6.4.11**(a)).
3. So $\{\ulcorner\sim \mathbf{P}\urcorner, \ulcorner\sim \mathbf{Q}\urcorner\} \subset \Gamma^\star$.
4. Now, $\{\ulcorner\sim \mathbf{P}\urcorner, \ulcorner\sim \mathbf{Q}\urcorner\} \vdash \ulcorner\sim (\mathbf{P} \vee \mathbf{Q})\urcorner$, in *SD* (proof below).
5. So $\ulcorner\sim (\mathbf{P} \vee \mathbf{Q})\urcorner \in \Gamma^\star$ (by 3, 4 and **6.4.9**).
6. So it is not the case that $\ulcorner\mathbf{P} \vee \mathbf{Q}\urcorner \in \Gamma^\star$ (again by **6.4.11**(a)).

So, if $\ulcorner\mathbf{P} \vee \mathbf{Q}\urcorner \in \Gamma^\star$, then $\mathbf{P} \in \Gamma^\star$ or $\mathbf{Q} \in \Gamma^\star$. Q.E.D.

Here is a proof of 4 (I think we did something very like this in class, but I do a derivation here anyway for completeness' sake).

1	$\sim \mathbf{P}$	A
2	$\sim \mathbf{Q}$	A
3	$\mathbf{P} \vee \mathbf{Q}$	A/$\sim$I
4	$\mathbf{P}$	A/$\vee$E
5	$\mathbf{P}$	4, R
6	$\mathbf{Q}$	A/$\vee$E
7	$\sim \mathbf{P}$	A/$\sim$E
8	$\mathbf{Q}$	6, R
9	$\sim \mathbf{Q}$	2, R
10	$\mathbf{P}$	7-9, $\sim$E
11	$\mathbf{P}$	3, 4-5, 6-10, $\vee$E
12	$\sim \mathbf{P}$	1, R
13	$\sim (\mathbf{P} \vee \mathbf{Q})$	3-12, $\sim$I

Now to prove the other direction: if $\mathbf{P} \in \Gamma^\star$ or $\mathbf{Q} \in \Gamma^\star$, then $\ulcorner\mathbf{P} \vee \mathbf{Q}\urcorner \in \Gamma^\star$. Again, I do this by proving the contrapositive.

1. Suppose it is not the case that $\ulcorner\mathbf{P} \vee \mathbf{Q}\urcorner \in \Gamma^\star$.
2. Then $\ulcorner\sim (\mathbf{P} \vee \mathbf{Q})\urcorner \in \Gamma^\star$ (by **6.4.11**(a)).
3. So $\{\ulcorner\sim (\mathbf{P} \vee \mathbf{Q})\urcorner\} \subset \Gamma^\star$.

4. Now, $\{\ulcorner\sim(\mathbf{P}\vee\mathbf{Q})\urcorner\}\vdash\ulcorner\sim\mathbf{P}\urcorner$ in *SD*, and $\{\ulcorner\sim(\mathbf{P}\vee\mathbf{Q})\urcorner\}\vdash\ulcorner\sim\mathbf{Q}\ulcorner$ in *SD* (proof below).

5. So, $\ulcorner\sim\mathbf{P}\urcorner\in\Gamma^\star$ and $\ulcorner\sim\mathbf{Q}\urcorner\in\Gamma^\star$ (by 3, 4 and **6.4.9**).

6. So it is not the case that either $\mathbf{P}\in\Gamma^\star$ or $\mathbf{Q}\in\Gamma^\star$ (by 6.4.11(a) again).

So, if $\mathbf{P}\in\Gamma^\star$ or $\mathbf{Q}\in\Gamma^\star$, then $\ulcorner\mathbf{P}\vee\mathbf{Q}\urcorner\in\Gamma^\star$. Q.E.D.

Here is a proof of the first half of 4 — i.e., that $\{\ulcorner\sim(\mathbf{P}\vee\mathbf{Q})\urcorner\}\vdash\ulcorner\sim\mathbf{P}\urcorner$.

1	$\sim(\mathbf{P}\vee\mathbf{Q})$	A
2	$\quad\mathbf{P}$	A/$\sim$I
3	$\quad\mathbf{P}\vee\mathbf{Q}$	2, $\vee$I
4	$\quad\sim(\mathbf{P}\vee\mathbf{Q})$	1, R
5	$\sim\mathbf{P}$	2-4, $\sim$I

The proof of the other half of 4 is the same, except you replace the '**P**'s on lines 2 and 5 with '**Q**'s.

So, I've proven that if $\ulcorner\mathbf{P}\vee\mathbf{Q}\urcorner\in\Gamma^\star$, then $\mathbf{P}\in\Gamma^\star$ or $\mathbf{Q}\in\Gamma^\star$, and I've proven that if $\mathbf{P}\in\Gamma^\star$ or $\mathbf{Q}\in\Gamma^\star$, then $\ulcorner\mathbf{P}\vee\mathbf{Q}\urcorner\in\Gamma^\star$. It follows that $\ulcorner\mathbf{P}\vee\mathbf{Q}\urcorner\in\Gamma^\star$ if and only if $\mathbf{P}\in\Gamma^\star$ or $\mathbf{Q}\in\Gamma^\star$. And that concludes the proof.

Question 2

We're trying to prove *Inductive Step*, on p. 273 of TLB, for the case in which **P**, a sentence containing $\mathbf{k}+1$ occurrences of connectives, has the form $\ulcorner\mathbf{Q}\vee\mathbf{R}\urcorner$.

1. Suppose that every sentence of *SL* with **k** or fewer occurrences of connectives is such that it is true on $\mathbf{A}^\star$ if and only if it is a member of $\Gamma^\star$ (i.e., suppose the antecedent of *Inductive Step*).

2. Now, $\ulcorner\mathbf{Q}\vee\mathbf{R}\urcorner$ is true on $\mathbf{A}^\star$ iff either **Q** is true on $\mathbf{A}^\star$ or **R** is true on $\mathbf{A}^\star$ (by definition of '$\vee$').

3. And **Q** is true on $\mathbf{A}^\star$ iff $\mathbf{Q}\in\Gamma^\star$, and **R** is true on $\mathbf{A}^\star$ iff $\mathbf{R}\in\Gamma^\star$ (by 1, and the fact that $\mathbf{Q},\mathbf{R}$ both contain **k** or fewer occurences of connectives).

4. So $\ulcorner\mathbf{Q}\vee\mathbf{R}\urcorner$ is true on $\mathbf{A}^\star$ iff either $\mathbf{Q}\in\Gamma^\star$ or $\mathbf{R}\in\Gamma^\star$ (from 2, 3).

5. So $\ulcorner\mathbf{Q}\vee\mathbf{R}\urcorner$ is true on $\mathbf{A}^\star$ if and only if $\ulcorner\mathbf{Q}\vee\mathbf{R}\urcorner\in\Gamma^\star$ (by **6.4.11**(c) — i.e., the thing we just proved in Question 1).

So *Inductive Step* is true for the case in which **P** has the form $\ulcorner\mathbf{Q}\vee\mathbf{R}\urcorner$. Q.E.D.

Question 3

The completeness proof for *SD* will fail, as a proof for the completeness of $SD^\star$, at the part where we try to prove **6.4.11**(b) — i.e., the proof that $\ulcorner \mathbf{P} \& \mathbf{Q} \urcorner \in \Gamma^\star$ if and only if both $\mathbf{P} \in \Gamma^\star$ and $\mathbf{Q} \in \Gamma^\star$ (where $\Gamma^\star$ is a maximal consistent-in-*SD* set of sentence of *SL*; **P**, **Q** are sentence of *SL*) will not go through. In particular, the proof that if $\ulcorner \mathbf{P} \& \mathbf{Q} \urcorner \in \Gamma^\star$ then both $\mathbf{P} \in \Gamma^\star$ and $\mathbf{Q} \in \Gamma^\star$ will not go through. Note that the proof of that part of **6.4.11**(b), on p. 272 of TLB, involves appealing to the Conjunction Elimination rule explicitly.

In fact, it will not, in general, be the case that a maximal consistent-in-$SD^\star$ set is such that if $\ulcorner \mathbf{P} \& \mathbf{Q} \urcorner \in \Gamma^\star$ then both $\mathbf{P} \in \Gamma^\star$ and $\mathbf{Q} \in \Gamma^\star$ (though this is quite hard to prove, and I don't do so here). There will, for example, be maximal consistent-in-$SD^\star$ sets that are supersets of {'$A \& B$', '$\sim A$'}.

Because the proof of **6.4.11**(b) fails, the proof of what the book calls the 'Consistency Lemma' fails too; in particular, case 2 of the inductive step fails. Even more in particular, the part of case 2 in which we prove that if $\ulcorner \mathbf{Q} \& \mathbf{R} \urcorner$ is false on $\mathbf{A}^\star$ than it is not in $\Gamma^\star$ will fail. That part of the proof relies on the part of **6.4.11**(b) that fails without Conjunction Elimination. And you can see why: for a set that is a maximal consistent-in-$SD^\star$ superset of {'$A \& B$', '$\sim A$'}, '$A \& B$' will be false on $\mathbf{A}^\star$, but it is in there anyway.

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