Subject 24.242. Logic II. Sample problems from the fourth homework, due April 8

Define the *restricted sentential calculus* (*RSC*) as follows: The atomic RSC sentences are "A_0," "A_1," "A_2,", and so on, and the RSC sentences are the atomic sentences and expressions formed from RSC sentences by one or more applications of the operations of forming conditionals (sentences of the form $(\phi \rightarrow \psi)$) and negations (sentences of the form $\sim\phi$). A *normal truth assignment* (*NTA*) is a functions $\mathfrak{I}$ taking the RSC sentences to $\{0,1\}$ that meets these conditions, for any ϕ and ψ: $\mathfrak{I}((\phi \rightarrow \psi)) = 1$ iff either $\mathfrak{I}(\phi) = 0$ or $\mathfrak{I}(\psi) = 1$ (or both); $\mathfrak{I}(\sim\phi) = 1$ iff $\mathfrak{A}(\phi) = 0$. An RSC sentence is a *tautology* iff it's assigned the value 1 by every NTA.

An *RSC derivation* is a finite sequence of sentences, each with an associated finite premise set of sentences, conforming to the following rules:

(PI) You may write ϕ with $\{\phi\}$ as its premise set.
(CP) If you've written ψ with Γ as its premise set, you may write $(\phi \rightarrow \psi)$ with $\Gamma \sim \{\phi\}$ as its premise set.
(MP) If you've written $(\phi \rightarrow \psi)$ with Γ as its premise set and you've written ϕ with Δ as its premise set, you may write ψ with $\Gamma \cup \Delta$ as its premise set.
(MT) If you've written $(\sim\phi \rightarrow \sim\psi)$ with Γ as its premise set and you've written ψ with Δ as its premise set, you may write ϕ with $\Gamma \cup \Delta$ as its premise set.

We encode the RSC numerically by associating with each RSC sentence ϕ a code number $\ulcorner\phi\urcorner$, according to the following stipulation:

$\ulcorner A_i \urcorner = \text{Pair}(1,i)$.
$\ulcorner(\phi \rightarrow \psi)\urcorner = \text{Triple}(2, \ulcorner\phi\urcorner, \ulcorner\psi\urcorner) = \text{Pair}(2,\text{Pair}(\ulcorner\phi\urcorner, \ulcorner\psi\urcorner))$
$\ulcorner{\sim}\phi\urcorner = \text{Pair}(3, \ulcorner\phi\urcorner)$

1. Give a derivation of of the following sentence from the empty set of premises:
(a) $((A_0 \rightarrow A_1) \rightarrow ((A_1 \rightarrow A_2) \rightarrow (A_0 \rightarrow A_2)))$

2. Show that the set of codes of RSC sentences in Σ. (You don't need to go all the way to primitive notation here; you may use reasonable abbreviations.)

Learned commentary. In Benson Mates's book, *Elementary Logic*, you can find a proof that an RSC sentence is derivable from the empty set iff it's a tautology. Putting this observation together with the results of problems 4 and 5, we can conclude that the set of code numbers of tautologies is Δ. This is a striking difference between sentential calculus and predicate calculus. *Church's Theorem,* which we'll prove shortly, tells that the set of valid sentences of the predicate calculus, though Σ, is not Δ.