Philosophy 244: #17—Intensional Objects

Intensional Objects

Entities obtained by stringing together "ordinary" objects, one per world and/or time, are called *intensional objects*. They give us a new and more charitable take on Converse Carban. If there has to be someone in charge of the Armed Forces at all times, there's an intensional entity *The Commander in Chief* who is always in charge of the Armed Forces. If the first bird on a given morning is the only one who can be sure to get a worm, then there's an elusive further character *the early bird* who always gets *the worm*. And so on.

More examples please!

Here's the problem: on the intensional object interpretation, which is what we're trying now, Converse Carban comes out valid. But it *shouldn't* if we are interested in contingent identity systems, for these were supposed to *weaker* than earlier systems. The idea was to step back from □I and whatever brings □I in its wake; nothing new was to be added. Something new *has* been added, though, for Converse Carban was not a theorem of those earlier systems.

This kind of trouble arises, it might be argued, because we're allowing intensional objects to be introduced at will; all imaginablle such objects, however gerrymandered or unintuitive, have allowed into the domain. Suppose we try to limit the construction somehow. An intensional object is a function *i* from worlds into the domain. A model is a quintuple <WRDIV> where I is a set of intensional objects. Variable assignments μ will now take variables not to domain elements but members of this set I, that is, to particular intensional objects *i*. Instead of $\mu(x,\mathrm{w})$ we will now have $\mu(x) = i =$ a function from worlds to regular objects. An x-alternative to μ will be a ρ taking every y distinct from x to the same intensional object as μ; $\rho(y)=\mu(y)$.

[Vφ"] $V_\mu(\varphi(x_1....x_n),\mathrm{w})=1$ iff $<\mu(\mathrm{x}_1)(\mathrm{w})....\mu(\mathrm{x}_n)(\mathrm{w}),\mathrm{w}>\epsilon V(\varphi)$
[V∀] $V_\mu(\forall x\alpha,\mathrm{w})=1$ iff $V_\rho(\alpha,\mathrm{w})=1$ for every x-alternative ρ of μ

A good portion of Carnap's *Meaning & Necessity* is devoted to defending this double-dealing, which he called "The Method of Extension and Intension." Quine called it "Carnap's curious double interpretation of the variables."

A lot of logicians have thought there is something funny in these two clauses. When we're interpreting *predicates*, variables contribute regular "extensional" objects, members of the domain: $\mu(x)(w) = i(\mathrm{w}) = \mathrm{o}$. When we're working with *quantifiers*, they range over intensional objects *i*, or functions from worlds to domain-elements. Shouldn't the predicates, including the identity predicate, apply to intensional objects too? Then however we'd be back where we started, with identities holding necessarily. How well our hopes for contingent identity are realized by this method thus seems open to question.

And not just to their values in the world in question?

Logical Systems

How is contingent identity to be dealt with axiomatically? The Leibniz axiom has got to be limited so that α is non-modal; in fact it suffices to apply it just to predicates:

I2" $x=y\supset(\varphi(x)\supset \varphi(y))$.

S+BF with the addition of I1 ($x=x$) and I2" will be S+CI. Soundness is easy. For completeness we again need a canonical model. What will play the role of I, the set of intensional objects? For each world w and variable x, let

$i_x(x)(\mathrm{w})$ = the first y such that $x=y\epsilon\mathrm{w}$.

I is the set of i_x for each variable x, and the canonical assignment μ takes each x to i_x. As usual, $\langle x_1 \ldots x_n, \mathrm{w}\rangle \epsilon \mathrm{V}(\varphi)$ iff $\varphi(x_1 \ldots x_n) \epsilon \mathrm{w}$.

Prop. 18.1 $\mathrm{V}_\mu(\alpha,\mathrm{w})=1$ iff $\alpha \epsilon \mathrm{w}$.

There's some flexibility in the *amount* of contingent identity/distinctness allowed. Say we want a semantics validating □I ($x=y \supset \Box x=y$) but not necessarily □NI ($x \neq y \supset \Box x \neq y$). Then we stipulate that when wRu, if i_1 and i_2 agree on w, they agree on u—while not stipulating that when wRu, if i_1 and i_2 agree on u, they agree on w.

Completeness follows as before. If α is S+CI-valid, then it holds in all worlds of all models built on S-frames. If the canonical model is built on an S-frame, then α holds in all worlds of the canonical model, that is, in all maximal S+CI-consistent sets. A statement belonging to all maximal consistent S+CI-sets is a theorem. Hence every valid α is a theorem, as promised.

Plenitude

If the intensional domain I consists of *all* intensional objects (all functions from W into D) then the converse Carban formula $\Box\exists x \varphi x \supset \exists x \Box \varphi x$ becomes valid. It is not any sort of contingent identity system we're dealing with, in that case, because CI systems are weaker than, say, S+BF+□I+□NI, and S+BF+□I+□NI doesn't have $\Box\exists x \varphi x \supset \exists x \Box \varphi x$ as a theorem.

Fine, so they're not contingent identity systems. The question stIll arises: is their logic, the logic of ALL intensional objects, axiomatizable? The answer in a word is NO, except under special conditions. Of the systems S we have considered, the logic of IOs based on S is unaxiomatizable with the single exception of S=S5.

A definition. Given a class $\mathcal{C}$ of frames, α is "$\mathcal{C}$-valid for intensional objects" iff $\mathrm{V}_\mu(\alpha,\mathrm{w})=1$ for every wϵW in every intensional object model <WRDV> (I is omitted since it's predictable from the rest) based on a frame $\langle \mathrm{WR} \rangle \epsilon \mathcal{C}$. The claim is that $\mathcal{C}$-validity thus defined is not for the most part axiomatizable.

Until further notice, $\mathcal{C}$-valid means $\mathcal{C}$-valid for intensional objects.

The method of proof is interesting. Remember second-order logic, the logic that lets us quantify into predicate position? There's a way to translate wffs of second-order logic into wffs of modal predicate logic so that validity is preserved. This is important because second-order logic is *unaxiomatizable*. If intensional modal logic were axiomatizable, with no restriction on the intensions, this would enable an axiomatization of second-order logic via the translation. Ba-boom! Contradiction! So the modal logic of all intensional objects is not axiomatizable.

We'll be working with just a fragment of second-order logic, but it too is unaxiomatizable.

The fragment $\mathcal{L}^2$ of 2nd order logic we need involves just (a) a bunch of one place predicate variables φ, ψ, etc. and (b) a two-place predicate constant *R*. Just as the *term* position in φx can be existentially generalized in 1st order logic to obtain $\exists x \varphi x$, the *predicate* position can be generalized in 2nd order logic to obtain $\exists^2 \varphi\ \varphi x$. A model for L^2 is just like a model for first-order logic except that, now that we're understanding φ, ψ, etc. as variables, their values are given by μ instead of V. For each one-place φ, $\mu(\varphi) \subset \mathrm{D}$. *R* is the only predicate constant; its value V(*R*) in a model <DV> is a subset of D×D. The only new semantical rule we will need is

[V$\forall^2$] $\mathrm{V}_\mu(\forall^2 \varphi\ \alpha)=1$ iff $\mathrm{V}_\rho(\alpha) = 1$ for every φ-alternative ρ of μ.

The next step is to reconceive the semantics of $\mathcal{L}^2$ so that it bears on modality. The domain D of $\mathcal{L}^2$ is conceived as a set W of worlds. The interpretation function V interprets R as the accessibility relation on D. To help us keep this in mind, we refer to D as W, and to domain-elements as w. Since all that V does is identify the accessibility relation R, a model for language L^2 is not essentially different from a frame for modal propositional logic. <D,V> "=" <W,R>.

Translation

Now we're getting somewhere. Just as we imagine the domain of $\mathcal{L}^2$ as a set of worlds, we can imagine the domain of our intensional object model as a set {0,1} of truth-values. A subset A of W (from our second-order non-modal model) may be coded by the intensional object i_A taking w to 1 iff wϵA and 0 if w$\notin$A.

i_A in the usual parlance is A's characteristic function.

That's the simple version anyway; it has to be complicated in two slightly mind-bending ways. First, the truth-values associated with 1 and 0 will be allowed to flip from world to world; 1 is sometimes truth and sometimes falsehood. Second, there may be more objects in D than just {0,1}; it can be any set of objects as long as they're suitably subdivided into the "true" and the "false." .

Every wff α of $\mathcal{L}^2$ is going to be translated into a wff $\tau(\alpha)$ of a language $\mathcal{L}^*$ of modal predicate logic. $\mathcal{L}^*$ has just a single one-place predicate T. T applies in a world w to the domain elements that will count as "true." What it takes for a world w to go into the set coded by i_A is that i_A(w)ϵV(T,w), that is, i_A takes w to the subset of its domain that plays the "true" role.

Now the translation; the individual variables of $\mathcal{L}^*$ are assumed to include all those of $\mathcal{L}^2$ plus an individual variable x_φ for every predicate variable φ of $\mathcal{L}^2$.

$\tau(\varphi x) = \Diamond(\mathrm{T}x \& \mathrm{T}x_\varphi)$
$\tau(x\mathrm{R}y) = \Diamond(\mathrm{T}x \& \Diamond \mathrm{T}y)$
$\tau(\neg\alpha) = \neg\tau(\alpha)$
$\tau(\alpha \vee \beta) = \tau(\alpha) \vee \tau(\beta)$
$\tau(\forall x \alpha) = \forall x \tau(\alpha)$
$\tau(\forall^2 \varphi\ \alpha) = \forall x_\varphi\ \tau(\alpha)$

The atomic cases are the strangest. Think of the first like this. If *x* in $\mathcal{L}^2$ is assigned a world *w*, then *x* in $\mathcal{L}^*$ is assigned an intensional object *i* that is "true" (that satisfies T) only in w. Likewise if φ in $\mathcal{L}^2$ is assigned a subset A of W, then x_φ is assigned in $\mathcal{L}^*$ an intensional object that is true (that satisfies T) in a world w iff wϵA.

Bearing all this in mind, when will $\Diamond(\mathrm{T}x \& \mathrm{T}x_\varphi)$ be true? It will be true iff a world is visible at which both T*x* and Tx_φ are true, iff a world is visible at which

(T*x*) the world that μ assigned to *x* in $\mathcal{L}^2$ and
(Tx_ϕ) a member of the set of worlds that μ assigned to φ in L^2.

....iff $V_\mu(\varphi x, x) = 1$. Likewise, $\Diamond(\mathrm{T}x \& \Diamond \mathrm{T}y)$ is true iff T*x* and $\Diamond$T*y* are true at some visible world, iff $\Diamond$T*y* is true at the world μ assigns to *x* in $\mathcal{L}^2$, iff the world μ assigns to *x* bears R to the world assigned to *y*, iff $V_\mu(x\mathrm{R}y) = 1$.

So the truth of $\Diamond(\mathrm{T}x \& \mathrm{T}x_\varphi)$ in the modal language goes with the truth of φx in the second-order language, and the truth of $\Diamond(\mathrm{T}x \& \Diamond \mathrm{T}y)$ goes with the truth of *xRy*.

Testing

Suppose you are given a frame <WR> and an intensional objects model <WRDV*> for $\mathcal{L}^*$ based on <WR> in which, for every wϵW, there are <u,w>ϵV*(T) and <v,w>$\notin$V*(T), that is, some domain elements are true and some false. μ for $\mathcal{L}^2$ and μ^* for $\mathcal{L}^*$ *correspond* iff for every wϵW and every variable x or φ of $\mathcal{L}^2$

(i) $\langle \mu^*(x)(\mathrm{w}), \mathrm{w} \rangle \epsilon \mathrm{V}^*(\mathrm{T})$ iff $\mathrm{w} = \mu(x)$
(ii) $\langle \mu^*(x_\varphi)(\mathrm{w}), \mathrm{w} \rangle \epsilon \mathrm{V}^*(\mathrm{T})$ iff $\mathrm{w} \epsilon \mu(\varphi)$.

Where V is the valuation of $\mathcal{L}^2$ that maps the predicate *R* to the relation R of our intensional objects model, and V_μ assigns to α the usual truth-value determined according to the rules of 2nd-order logic,we have:

Prop. 18.2: Suppose α is a wff of L^2, wϵW, wR$\mu(x)$ for every variable x, and μ and μ^* correspond. Then $V_\mu(\alpha) = V_{\mu*}{}^*(\tau(\alpha),w)$.

Proof: See 338-340

Next we have to link up validity on a frame with validity in the corresponding intensional objects model. Define

Wx =df ◇Tx&∀y(□(Tx⊃Ty)∨□(Tx⊃¬Ty))

Lemma 18.3 $V_{\mu*}{}^*$(Wx,w*)=1 iff w* sees a w s.t. <$\mu^*(x)$(w'),w'>ϵV*(T) iff w'=w.

For all w'.

This considered as a condition on μ^* is just what it takes for μ^* to correspond with some μ, since we can just let $\mu(x)$ be the unique w such that <$\mu^*(x)$(w),w>ϵV*(T).

Prop. 18.4 If $\alpha(x_1\text{-}x_n)$ is an L^2 wff, and α^* = □(∃xTx&∃x¬Tx)&Wx_1&...Wx_n)⊃$\tau(\alpha)$, then α is valid on <WR> iff α^* is valid on every IO model based on <WR>.

Unaxiomatizability

Now we are almost there. For a system to be axiomatizable is for there to be an effective means of enumerating its theorems. If IO systems were axiomatizable then we could enumerate the valid wffs of L^2 by enumerating the wffs valid on every IO model, and putting α on the list of L^2 theorems whenever the enumeration hits α^*.

Drilling down a bit: Suppose that S is a propositional modal system some of whose frames <WR> have a number-like structure. W is countable, and R is linear (of any two worlds, exactly one bears R to the other), transitive, and discrete (each w Rs a w' s.t. there is no w" between them). Then we will be able to define successor in terms of R: y is the successor of x iff xRy and there is no z such that xRzRy. And there will be statements of L^2 that via this definition express 2nd-order Peano's axioms, and that will truly characterize the S-frames with a number-like structure.

Let the conjunction of these Peano statements be PA^2. The statements following from PA^2 in 2nd-order logic are precisely the arithmetical truths. The arithmetical truths are very far from being effectively enumerable. But they would have to be if S+IO was effectively enumerable, that is, {α^* | α^* is an L* sentence valid on all IO-models built on S-frames} were effectively enumerable. For then by 18.4, {α | α is 2nd-order valid on any <DV> that's an S-frame} would be effectively enumerable. But then the following would be effectively enumerable, each because of the one before.

{β | AX⊃β is 2nd-order valid on any <DV> that's an S-frame}
{β | AX⊃β is 2nd-order valid}
{β | β is an arithmetical truth}

There's an infinite hierarchy of increasingly complex sets, with the effectively decidable sets and then the effectively enumerable one. Y is the next level up from X iff it can be decided by a Turing machine with an oracle for X. The arithmetical truths are more complex than anything on this hierarchy.

But again it's a known fact, the "deep fact" behind Godel's theorem, that the set of arithmetical truths is the furthest thing from effectively enumerable.

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