Oct. 31 2005: Lecture 16:

Integral Theorems

Reading:
Kreyszig Sections: §9.8 (pp:510–14) , §9.9 (pp:515–20)

Higher-dimensional Integrals

The fundamental theorem of calculus was generalized in a previous lecture from an integral over a single variable to an integration over a region in the plane. Specifically, for generalizing to Green's theorem in the plane, a vector derivative of a function integrated over a line and evaluated at its endpoints was generalized to a vector derivative of a function integrated over the plane.

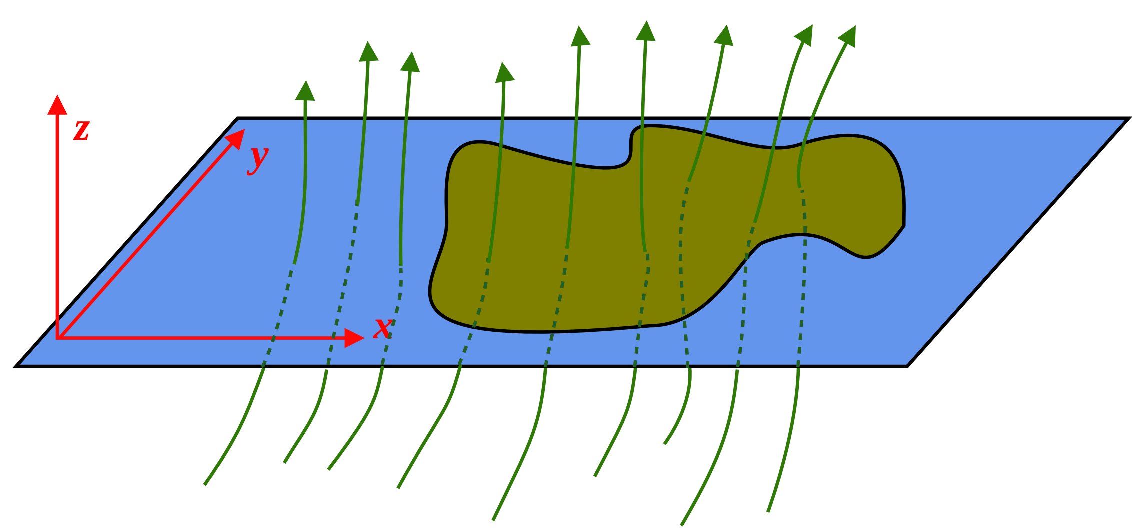

Figure 16-1: Illustrating how Green's theorem in the plane works. If a *known* vector function is integrated over a region in the plane then that integral should only depend on the bounding curve of that region.

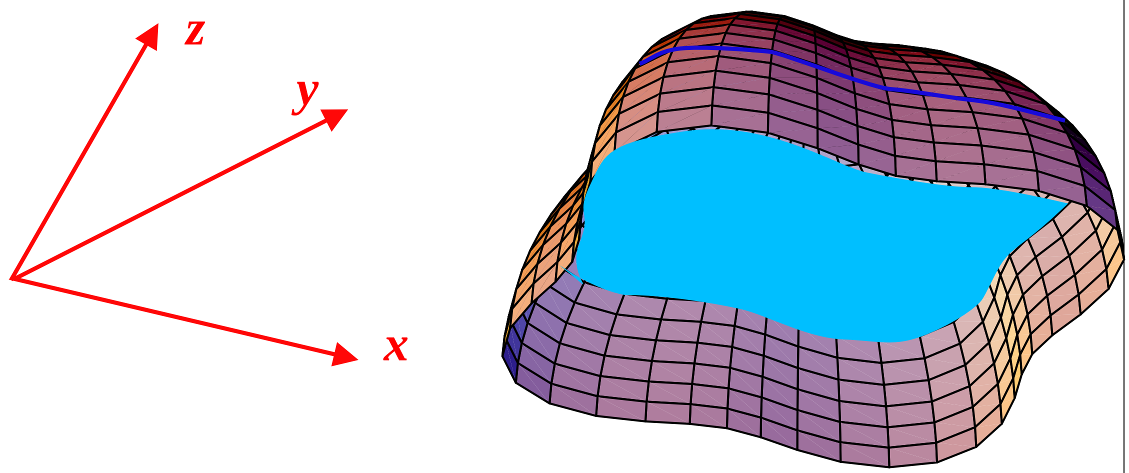

Figure 16-2: Illustration of a generalization to the Green's theorem in the plane: Suppose there is a bowl of a known shape submerged in a fluid with a trapped bubble. The bubble is bounded by two different surfaces, the bowl down to $z = 0$ and the planar liquid surface at that height. Integrating the function $\int_{V_B} dV$ over the bubble gives its volume. The volume must also be equal to an integral $\int\int_{\partial V_B} z dx dy$ over the (oriented) surface of the liquid. However, the volume of bubble can be determined from only the curve defined by the intersection of the bowl and the planar liquid surface; so the volume must also be equal to $\oint_C$(some function)ds.

The Divergence Theorem

Suppose there is "stuff" flowing from place to place in three dimensions.

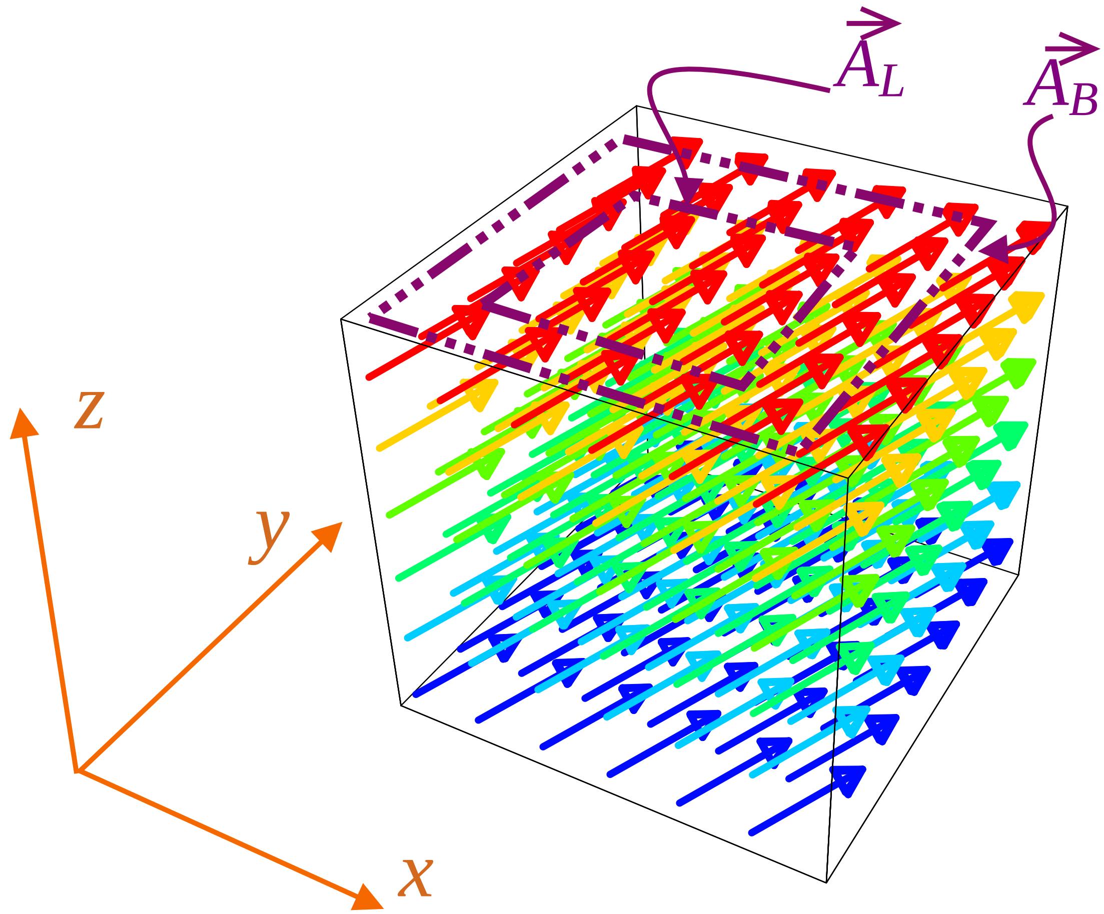

Figure 16-3: Illustration of a vector "flow field" $\vec{J}$ near a point in three dimensional space. If each vector represents the rate of "stuff" flowing per unit area of a plane that is normal to the direction of flow, then the dot product of the flow field integrated over a planar oriented area $\vec{A}$ is the rate of "stuff" flowing through that plane. For example, consider the two areas indicated with purple (or dashed) lines. The rate of "stuff" flowing through those regions is $\vec{J} \cdot \vec{A_B} = \vec{J} \cdot \hat{k} A_B$ and $\vec{J} \cdot \vec{A_L} = \vec{J} \cdot \hat{k} A_L$.

If there are no sources or sinks that create or destroy stuff inside a small box surrounding a point, then the change in the amount of stuff in the volume of the box must be related to some integral over the box's surface:

$$\begin{aligned}
\frac{d}{dt}(\text{amount of stuff in box}) &= \frac{d}{dt}\int_{\text{box}} (\frac{\text{amount of stuff}}{\text{volume}}) dV \\
&= \int_{\text{box}} \frac{d}{dt}(\frac{\text{amount of stuff}}{\text{volume}}) dV \\
&= \int_{\text{box}} (\text{some scalar function related to } \vec{J}) dV \\
&= \int_{\substack{\text{box}\\ \text{surface}}} \vec{J} \cdot d\vec{A}
\end{aligned} \tag{16-1}$$

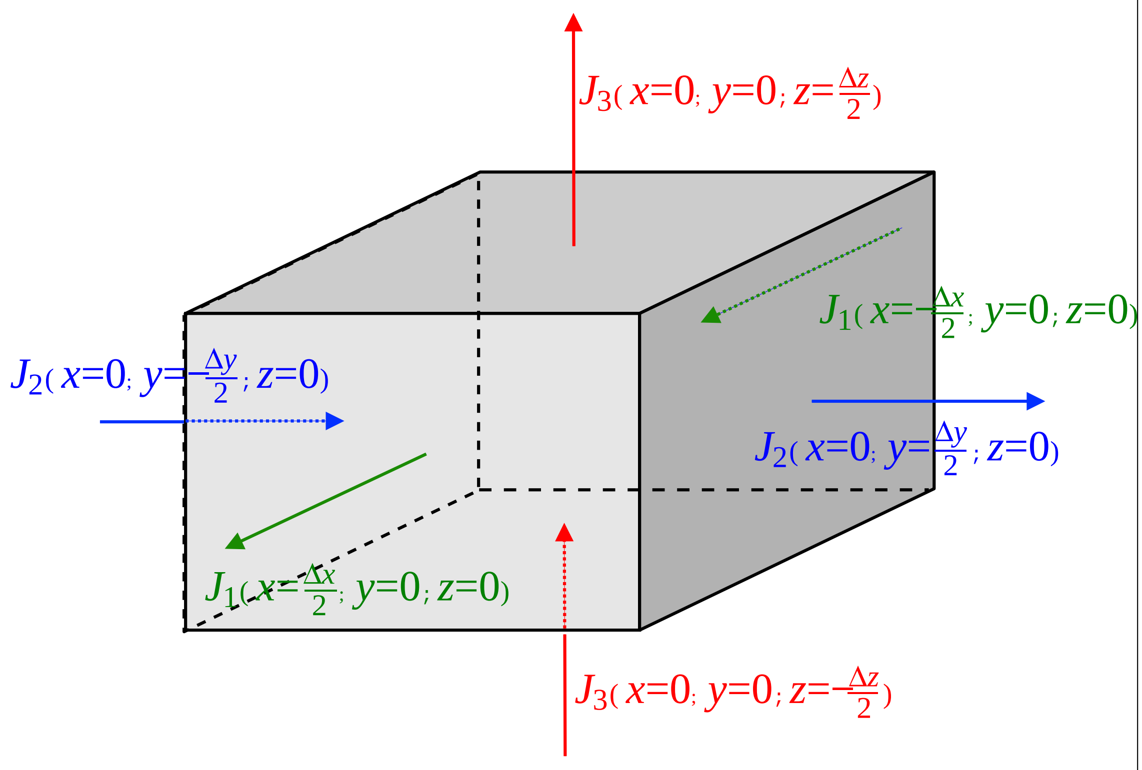

Figure 16-4: Integration of a vector function near a point and its relation to the change in that vector function. The rate of change of stuff is the integral of flux over the outside—and in the limit as the box size goes to zero, the rate of change of the amount of stuff is related to the sum of derivatives of the flux components at that point.

To relate the rate at which "stuff M" is flowing into a small box of volume $\delta V = dxdydz$ located at (x, y, z) due to a flux $\vec{J}$, note that the amount that M changes in a time Δt is:

$$\begin{aligned}\Delta M(\delta V) &= (M \text{ flowing out of } \delta V) - (M \text{ flowing in } \delta V) \\ &= \Big[\vec{J}(x - \tfrac{dx}{2})\hat{i}dydz - \vec{J}(x + \tfrac{dx}{2}) \cdot \hat{i}dydz \\ &+ \vec{J}(y - \tfrac{dy}{2})\hat{j}dzdx - \vec{J}(y + \tfrac{dy}{2}) \cdot \hat{j}dzdx \\ &+ \vec{J}(z - \tfrac{dz}{2})\hat{k}dxdy - \vec{J}(z + \tfrac{dz}{2}) \cdot \hat{k}dxdy \Big] \Delta t \\ &= -(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z})\delta V \Delta t + \mathcal{O}(dx^4)\end{aligned} \tag{16-2}$$

If $C(x, y, z) = M(\delta V)/\delta V$ is the concentration (i.e., stuff per volume) at (x, y, z), then in the limit of small volumes and short times:

$$\frac{\partial C}{\partial t} = -(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}) = -\nabla \cdot \vec{J} = -\text{div}\vec{J} \tag{16-3}$$

For an arbitrary closed volume V bounded by an oriented surface ∂V:

$$\frac{dM}{dt} = \frac{d}{dt}\int_V C dV = \int_V \frac{\partial C}{\partial t} dV = -\int_V \nabla \cdot \vec{J} dV = -\int_{\partial V} \vec{J} \cdot d\vec{A} \tag{16-4}$$

The last equality

$$\int_V \nabla \cdot \vec{J} dV = \int_{\partial V} \vec{J} \cdot d\vec{A} \tag{16-5}$$

is called the Gauss or the divergence theorem.

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Hamaker Interaction between a point and Closed Volume

Calculating the Van der Walls potential (also called London Dispersion potential) of a point-particle the vicinity of a finite cylinder. The interaction energy due two induced dipoles, one located at $\vec{r} = (\xi, \eta, \zeta)$ and another located at $\vec{x} = (x, y, z)$ goes like

$$\frac{-1}{\|\vec{r} - \vec{x}\|^6} \tag{16-6}$$

Integrating this function for $\vec{r}$ ranging over the volume of a cylinder of length L and radius R will give the potential for a point particle located at $\vec{x}$ due to the entire cylinder. This integration has no simple closed form, so a numerical integration is necessary. The following method, using the divergence theorem, makes the numerical integration more efficient by converting a volume integral to a surface integral of a vector potential.

Integrating a vector potential over a surface

Stokes' Theorem

The final generalization of the fundamental theorem of calculus is the relation between a vector function integrated over an oriented surface and another vector function integrated over the closed curve that bounds the surface.

A simplified version of Stokes's theorem has already been discussed—Green's theorem in the plane can be written in full vector form:

$$\begin{aligned}\int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy &= \int_R \nabla \times \vec{F} \cdot d\vec{A} \\ = \oint_{\partial R} (F_1 dx + F_2 dy) &= \oint_{\partial R} \vec{F} \cdot \frac{d\vec{r}}{ds} ds\end{aligned} \tag{16-7}$$

as long as the region R lies entirely in the $z =$ constant plane.

In fact, Stokes's theorem is the same as the full vector form in Eq. 16-7 with R generalized to an oriented surface embedded in three-dimensional space:

$$\int_R \nabla \times \vec{F} \cdot d\vec{A} = \oint_{\partial R} \vec{F} \cdot \frac{d\vec{r}}{ds} ds \tag{16-8}$$

Plausibility for the theorem can be obtained from Figures 16-1 and 16-2. The curl of the vector field summed over a surface "spills out" from the surface by an amount equal to the vector field itself integrated over the boundary of the surface. In other words, if a vector field can be specified everywhere for a *fixed* surface, then its integral should only depend on some vector function integrated over the boundary of the surface.

Maxwell's equations

The divergence theorem and Stokes's theorem are generalizations of integration that invoke the divergence and curl operations on vectors. A familiar vector field is the electromagnetic field and Maxwell's equations depend on these vector derivatives as well:

$$\begin{aligned}\nabla \cdot \vec{B} &= 0 \qquad & \nabla \times \vec{E} &= \frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{j} \qquad & \nabla \cdot \vec{D} &= \rho\end{aligned} \tag{16-9}$$

in MKS units and the total electric displacement $\vec{D}$ is related to the total polarization $\vec{P}$ and the electric field $\vec{E}$ through:

$$\vec{D} = \vec{P} + \epsilon_o \vec{E} \tag{16-10}$$

where ϵ_o is the dielectric permittivity of vacuum. The total magnetic induction $\vec{B}$ is related to the induced magnetic field $\vec{H}$ and the material magnetization through

$$\vec{B} = \mu_o(\vec{H} + \vec{M}) \tag{16-11}$$

where μ_o is the magnetic permeability of vacuum.

Ampere's Law

Ampere's law that relates the magnetic field lines that surround a static current is a macroscopic version of the (static) Maxwell equation $\nabla \times \vec{H} = \vec{j}$:

Gauss' Law

Gauss' law relates the electric field lines that exit a closed surface to the total charge contained within the volume bounded by the surface. Gauss' law is a macroscopic version of the Maxwell equation $\nabla \cdot \vec{D} = \rho$: