Plane Stress vs. Plane Strain

Hooke's Law for isotropic materials:

$$\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \epsilon_{23}\\ \epsilon_{31}\\ \epsilon_{12}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1+\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\nu\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix}$$

Case 1: Plane Stress

$\sigma_{33} = 0$, $\epsilon_{33} \neq 0$: $\epsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}(\sigma_{22} + \cancel{\sigma_{33}}^{\,0})$

$$\epsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22})$$

$$\sigma_{11} - \nu\sigma_{22} = E\epsilon_{11}$$

$$\boxed{\begin{aligned} E' &= E \\ \nu' &= \nu \end{aligned}}$$ For plane stress (i.e. thin sheet)

Case 2: Plane Strain

$$\epsilon_{33} = 0, \quad \sigma_{33} \neq 0$$

$$\epsilon_{33} = \frac{\sigma_{33}}{E} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22}) = 0 \rightarrow \sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

$$\epsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}(\sigma_{22} + \sigma_{33})$$

$$\epsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}\left(\sigma_{22} + \nu[\sigma_{11} + \sigma_{22}]\right)$$

$$\epsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu^2}{E}\sigma_{11} - \frac{\nu\sigma_{22}}{E}(1+\nu) = \frac{\sigma_{11}}{E}(1-\nu^2) - \frac{\nu\sigma_{22}}{E}(1+\nu)$$

$$\epsilon_{11} = \frac{1-\nu^2}{E}\left[\sigma_{11} - \sigma_{22}\frac{\nu}{1-\nu}\right] \rightarrow \sigma_{11} - \frac{\nu}{1-\nu}\sigma_{22} = \frac{E}{1-\nu^2}\epsilon_{11}$$

$$\boxed{\begin{aligned} E' &= \frac{E}{1-\nu^2} \\ \nu' &= \frac{\nu}{1-\nu} \end{aligned}}$$ For plane strain (i.e. thick plate)