PROFESSOR: OK, why don't we you get started again? I can understand why there is an air of excitement in the room since tomorrow's a holiday. But we still got two hours before the day is over. Any questions on what we have done up to this point? No, that's good.

Let me erase some of this art work then and ask why one would indulge in this rather bizarre exercise of taking the elements of something that represents a physical property and constructing a surface out of them. Well, the name suggests that maybe these surfaces, be they hyperboloids of one or two sheets or imaginary ellipsoids, have something to say about the property that the tensor that formed the coefficient of these functions is doing.

So let me take the case of an ellipsoid. That would be the case where all the coefficients are positive. And let's let this be x1 and this x2. And let me ask a first question.

If this is to be a well deserved function, does the surface transform in exactly the same way as the tensor elements? In other words, if we take the tensor in one coordinate system and then change the coordinate system to a different coordinate system, are the coefficients in front of x1, x2, and x3 still the elements of the tensor in that new coordinate system?

So let me show you that this is, in fact, the case. Suppose our original relation, sticking to conductivity, is that x sigma ij xi xj equals 1. And then for some reason or another, we change axes. So the new equation will be some different coefficients. Because as we change axes, they're going to wink on and off, still the same surface, but now referred to different axes xi prime and xj prime still equal to 1.

And let me now use the reverse transformation to put xi prime in the terms of xi. If we do that, we can say that xi prime is cli x sub l prime. That's the reverse transformation. So notice the inverted order of the direction cosine subscripts. And then xj prime is going to be equal to cmj times x sub ep prime.

To find this, let's just rearrange these terms in a trivial fashion. Then I will have cx sigma ij cil cmj times xl prime xm prime equals 1. And what is this? This is a summation of direction cosines where the first subscript goes with the subscript on sigma prime. What did I do wrong? This is mj. Why is that not coming first?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Well, what I would like to get this in the form of is a summation of sigma ij prime over two dummy indices l and m. What did I do wrong? Ah, yes, right, right, right, right, this is li. Thank you somebody said it. And my nose was right in it. And I didn't notice it.

OK, this is a summation of all the original elements sigma ij prime times dummy indices l and m. And this actually is sigma ijn. So it's the same set of coefficients. And that is a technicality, which probably you wouldn't want to worry about.

So anyway, the elements of the tensor transform in the same way that the equations for the surfaces transform. So if you change coordinate system, the coefficients that you should use should be the elements that are in the transform tensor.

OK, now I'm going to ask a question. Suppose I define some direction by a set of direction cosines li and I ask the value of the radius of the surface in that direction. So this is some radius vector. The radius vector will have components, R1, R2, R3 or Ri. And each of those components will be equal to the magnitude of R times the direction cosines li. Yes, sir?

AUDIENCE: Could you briefly rego over what you just stated before you erased it [INAUDIBLE]?

PROFESSOR: Oh, OK. I'm sorry. I said aij xi xj equals 1 is the original equation for the quadric. If we change the coordinate system, we're going to have some new elements aij prime times xi prime xj prime. Because we've got a new coordinate system. And therefore, the coefficients in front of the equation for the quadric have to change.

And now what I did was to express xi prime and xj prime in terms of xi and xj using the reverse transformation. And xi prime in terms of the original x's will be c sub l sub i x sub l prime. Usually, we write it xl equals cil x sub l. This is the reverse transformation where the order of the subscripts is reversed.

So this will give me x of i prime. The expression for x sub j prime will be given by cmj x sub j where m is a dummy index. And that's equal to 1.

And if I just rearrange this, then I will have cli cmj times aij prime equals 1 times xl xj. And this is going to be alx lxm. And this will be alm times xl xm equals 1, which is the equation for the quadric in the original coordinate system. I think that was better when I was standing in front of it so you couldn't see it.

It's just that the surface transforms formally to the surface that we would create if we used the new tensor elements for the same change of coordinates system, which you would probably will be willing to take my word for.

AUDIENCE: [INAUDIBLE]

PROFESSOR: This is a direction cosine for the--

AUDIENCE: [INAUDIBLE] right below.

PROFESSOR: Right below?

AUDIENCE: [INAUDIBLE]

PROFESSOR: cli cmj aij prime xl xm, and then I collected the cli cmj times alm. And that is the transfer. Yes?

AUDIENCE: In the previous [INAUDIBLE], you defined the derivitave side as being xi prime equals cil xl. [INAUDIBLE]?

PROFESSOR: You can say that x prime is cil times x sub l. And if we want to use the same direction cosine scheme but do it in reverse, we would say that x sub l is equal to c-- yeah, OK, x sub i is cli times x sub i, right, no, times x sub l-- is that right-- x sub l prime.

AUDIENCE: On the second line to the third line, you say that xi prime equals cli [INAUDIBLE]. It's either xi equals xcli xl prime or xi prime equals cil.

PROFESSOR: OK, you want to say this. xi prime and cil becomes xl, yes, yeah. And then mj times x-- that's x sub m. OK, and then this says that cil cjm times aij prime should be alm, right, OK, OK. So it's OK then. OK, that was supposed to be a small point that everybody would accept. But now it's done correctly.

OK, let's get back to this, which is more hair raising and exciting. What is the radius of the quadric in a given direction that we specify by three direction cosines l1, l2, l3?

So there's some radius from the center of the quadric out to the surface in that particular direction specified by three direction cosines. And we know what those three components are, sub i, are going to be in terms of the direction cosines, magnitude of R times li.

And those points at the terminus of the radius vector go to one point on the surface of the quadric. So these values of R are coordinates that satisfy the surface of the quadric. So we can say that just substitute Ri for the different values xi in the equation for the quadric.

And this says that sigma ij magnitude of R times li, that would correspond to x sub i times magnitude of R times l sub j. That's the magnitude of xj. That should be equal to 1.

And if I rearranged this slightly, this says that the magnitude of R squared is going to be equal to 1 over sigma ij times li lj. So the radius of the quadric gives me not the value of the property in that direction. This is the value of the property that will occur in the direction specified by the direction cosines lj. The radius of the quadric is going to be equal to 1 over the square root of the value of the property in that direction.

So this is why it's called the representation quadric. If you construct the surface from the tensor elements, you will have defined a quadratic form, which has the property that, as you go in different directions look at the distance out to the surface of the quadric in that direction, that that distance squared is going to be equal to 1 over the property in that direction or, alternatively, the radius is going to be 1 over the square root of the value of the property.

There is an enormous implication here. This says that the value of the property, if the quadratic form is an ellipsoid, the value of the property as we go around in different directions in the crystal is going to be a smooth, uniformly varying function. They're going to be no lobes sticking out. They're going to be no dimples, no lumps.

It's going to be an almost monotonous property, not very interesting. It's going to change in a uniform way. And in fact, we could put the two surfaces side by side.

If this is the value of the quadric as a function of direction, if we make a polar plot of the property as a function of direction, it's going to look sort of like the reciprocal of this. The minimum value of the property will be in the direction of the maximum value of the radius of the quadric. And the maximum value of the property is going to go in the direction of the minimum value.

So the value of the property as a function of direction is going to be a quasi-ellipsoidal sort of variation but not really an ellipsoid. It's going to go as this inverse square of the radii in ellipsoid.

But the thing is it's going to be something that varies uniformly between extreme values of the maximum and minimum value of the property. We are, I assure you, for higher ranked tensor property going to look at some absolutely wild surfaces with surfaces that do have lobes and extreme values and very, very irregular variation of properties with directions but not for second ranked tensor properties.

There will be a few variations on this theme, which are also interesting to touch upon. In principle, we can get other quadratic forms from the tensor if some of the elements of the tensor are negative. And in particular, what would we see-- and I'll draw this relative to the principal axes of the surface-- what would we see for an hyperboloid of one sheet?

This is the sort of surface that might result if one principal value of the tensor had a negative value. Well, this is the quadric. And this is a radius in one particular direction. What would this say about the property?

Well, let me, to make it clear, look at one of these sections through the quadric that our hyperbola. In directions like this, we have a radius that is a minimum out to the surface. In other directions, the radius gets progressively larger.

And then if we plot the reciprocal of the square root of that, that says that, in this direction, we get the maximum value of the property. And as the direction approaches the asymptote, the reciprocal will go down to 0.

But how did I get two y-axes here? OK, so the maximum will be in the direction of the minimum radius. And then it will go down to 0 for the asymptote. And that's going to be symmetrical on either side of this principle axis.

So what then are the radii in directions outside of the asymptotes of this hyperboloid? Within this range, the radius is imaginary. But if you square it, you get a negative number.

So within these two lobes, the value of the property is positive. As you go away from the asymptotes, you'll get another lobe like this and another lobe like this where the value of the property is negative.

How in the world could you get anything that looked like that? Well, a good example of a property that has this behavior is thermal expansion. And let me give you two examples.

The structure of selenium and tellurium is a hexagonal structure. And there are chain-like molecules that are pairs of bonds that rise up around a threefold screw axis. So this is two coordinated atoms in the structure just spirals up in this triangular spiral around the threefold screw axis.

So this is a material in which the bonding is very, very anisotropic. The bonding within these covalently bonded spirals, which are like springs that you might put on your screen door in the summertime, the bonding is very strong. Between these individual molecular chains, the bonding is very weak.

So what happens when you heat this stuff up? It expands like the dickens in the direction of these weak bonds. But the spirals are, in part, held in this extended form by repulsive interactions in like chains. So when these spirals move apart as a result of large thermal expansion in the plane normal to the chain, the chains relax a little bit.

So you have a large positive thermal expansion here. But in one direction along the normal to the hexagonal in the structure, the thermal expansion is negative. And that gives you a variation of property with direction that looks exactly like this. Negative value of the property, that means the structure, contracts in that direction, positive values this way. The structure expands. So there's a good example of this.

There's another example of a material, which, again, has a negative thermal expansion in one direction. And this is calcium carbonate, which is a complicated hexagonal structure. But it looks very, very much like the structure of rock salt in a distorted form.

These are the calcium atoms. And calcite is CaCL3 And the calciums are in a face-centered cubic arrangement just like in rock salt. The carbonate groups are very tightly bonded little triangles with the carbon in the middle.

And these things are arranged on the edges of the cell. And this is going to be very schematic. These triangles are normal to the body diagonal of the cells. So you have one family of triangles that are all parallel to one another. On the next edge, the triangles are anti-parallel to this first orientation.

So think of rock salt. Tip it up on its body diagonal. Every place you have a chlorine anion, place a triangle in an orientation that is perpendicular to the body's diagonal of the rock salt structure.

Again, these tightly bonded little triangles don't do much as you increase temperature. But the bonding between the calcium and the triangles is rather weak. So again, you find that the structure expands in one direction so strongly that it contracts to the other direction so calcium carbonate. The calcite form also has the distinction of having a negative thermal expansion coefficient.

We'll talk quite a bit about expansion coefficients as one example of a tensor property. Interestingly, and we'll say more about this and I'll give you some references when we come to this point, can you get a material that has a negative thermal expansion coefficient in all directions?

No, that seems as though it would violate in a flagrant way some vast law of thermodynamics that things have to increase their volume when you increase temperature. There were, however, a few odd ball materials that over a very, very limited temperature range would contract but only by a tiny amount in all directions.

And then one of the very interesting discoveries of recent years done primarily by a crystal chemist named Art Slate, who's out at the University of Oregon, he discovered a family of materials that have large negative expansion coefficients in all directions over a considerable range of temperatures, like 100 degrees. So these are materials, when you heat them up, they contract amazing as that seems.

And there's a structural reason for it. These are tetrahedral frameworks in which one corner of a tetrahedron is dangling. And as you heat it up, this tetrahedral corner can get closer to other tetrahedra, which takes energy to do. But in so doing, you actually are changing the net volume occupied by the solid so very, very anomalous set of compounds.

More of these have been found. There are probably a dozen materials now that have negative thermal expansion coefficients in all directions. So is the imaginary ellipsoid a viable representation quadric for real materials? Yes, in the case of thermal expansion coefficients.

OK, so the representation quadric then is a dandy device for seeing with one function how a property will vary with direction. For an ellipsodial quadric, the variation of the property itself with direction is not really ellipsodial but quasi-ellipsodial in that there's a small principal axis, a large principal axis. And the large value of the property goes with the short axis of the quadric.

So we have a nice device for representing the value of the property as a function of direction. It looks as though we've lost all information about the direction of the resulting vector, the generalized displacement. It looks as though that's not in here at all.

Well, there was an ad years ago for a bottle spaghetti sauce, which offends many cooks who are very fiercely proud of their own spaghetti sauce. So here's a wife who's using the canned stuff. And her husband comes home and looks at it very, very skeptically and says, where's the basil? And the housewife says, it's in there. It's in there. And he sniffs again and says, where is the basil? And she says finally, it's in there. It's in there.

Well, this is a similar situation. Where is the direction of the generalized force? And I say, it's in there. It's in there. Not obvious, but it's in there. So let me now show you where it is lurking. I think we have time to carry this through.

Let me describe how it's embodied in the quadric and then prove to you that this is indeed the case. And I'll use a general quadric in the form of an ellipsoid. That looks more like an egg than an ellipsoid. It has a pointed end.

OK, this is something that's called the radius normal property. And what it says, in words, is that, if you pick a particular direction relative to the quadric, we know that its length is going to be inversely proportional to the square root of the value of the property. If you want to know what the direction of the resulting vector is, for example, in our relation for conductivity now getting very warm, it says the current flow is given by a linear combination of every component of the electric field.

The radius normal properties is that, if you want to know where J is for this particular direction, look at the point where the radius vector intersects the surface of the quadric and, at that point, construct a perpendicular to the surface, which, in general, will not be parallel to R. And this will be the direction of J. It won't give you the magnitude. But it'll give you the direction of it.

OK, let me now prove to you that the quadric does have this property. In our conductivity relation, the direction of J is going to be given by the tensor relation. Namely that J sub i is equal to sigma ij times E sub j, which can be written as sigma ij times the direction cosines of E sub j times the magnitude of E.

I'm going to want to compare this expression, with which you're very familiar term by term, with the normal to the surface that we can compute from its derivative at that point. So this will say that J1 is equal to sigma 1, 1 l1 times the magnitude of E plus sigma 1, 2 times l2 times the magnitude of E plus sigma 1, 3 times l3 times the magnitude of E. And I don't need to write much more. J2 will be sigma 2, 1 times l1 magnitude of E plus sigma 2, 2 l2 times the magnitude of E and so on.

What is going to be the normal to the surface as a function of direction? OK, to do this I'm going to say that the normal to the surface is going to be, if I have some function of xyz, the normal to that function is the gradient. Let me say that that's G of xyz.

And we can say then that the x1 component of the normal is not going to be equal to but proportional to the gradient of the equation for the quadric with respect to x1, dx2, and dx3. So it's going to be proportional to the differential of the function that gives us the surface with respect to x1 times i plus the differential of the function with respect to x2 times j plus dF dx3 times k. So this is the normal entirely.

So if we split this into components, N1 is going to be equal to dF dx1. And if I differentiate the equation for the quadric, that's going to be 2 sigma 1, 1 times x1. Remember the equation for the quadrant is sigma 1, 1 times x1 squared plus sigma 1, 2 times x1 x2. So if I differentiate-- let me write it down.

If I take those terms and differentiate with respect to x1, I'm going to get 2 sigma 1, 1 times x1. Here I'll get sigma 1, 2 times x2. But then down in this line where I have a sigma 2, 1 x2 x1, if I differentiate with respect to x1, I'll have another term sigma 2, 1.

And if I differentiate with respect to x3, I'll have sigma 1, 3 plus sigma 3, 1 times x3. And a similar thing for the x2 component of the normal, that's going to be proportional to the gradient with respect to x2. And that's going to be equal to sigma 1, 2 plus sigma 2, 1 times x1 plus 2 sigma 2, 2 times x2 plus sigma 2, 3 plus sigma 3, 2 times x3 and similarly for N3.

OK, I've made my case if I can demonstrate that each component of J, J sub i, is proportional to each component of the gradient to the function that defines the quadric. If you look at them, they're close but no cigar. The first term is OK. I've got a 2 out in front of sigma 1, 1 for the x1 term.

Here, though, I have the sum of two off-diagonal terms. And for this expression, I have just sigma 1, 2. And down here I've just the single term sigma 2, 1. So the conclusion we're forced to draw is that these two expressions would have components of a vector that are parallel to one another if sigma 1, 2 was equal to sigma 2, 1.

Because if these were equal, I could write just twice sigma 1, 2. And down here I could write twice sigma 2, 1 and do that all the way through these two expressions. And then this factor of 2 would just be part of the proportionality constant.

So the radius normal property will be true or valid only for symmetric tensors. That is it's going to be true only if sigma ij is equal to sigma ji.

We mentioned when we first started talking about second ranked tensors that a tensor does not have to be symmetric. And showing that the tensor is symmetric has nothing to do with symmetry. Because it's symmetric in an algebraic sense not in the literal sense of geometrical symmetry.

And there are some properties, the thermal electricity tensor is one notable example, where the tensor is a second ranked tensor but it decidedly is not symmetric. So this is OK for most properties but not all.

OK, so here are properties of the representation surface that we can construct from the representation quadric. And what I would like to do following this and after the inevitable unpleasantness of a quiz is to look at some specific properties that illustrate second ranked tensor properties.

And in particular, I would like to look at some other sorts of second ranked tensor properties that represent generalized forces, namely stress and strain. You've seen these before in other contexts perhaps not actually defined rigorously in terms of tensors. Because, obviously, stress and strain can be regarded as generalized forces.

A stress is something that you can apply to a solid. And that will cause various things to happen. In addition to mechanical behavior, different properties and effects can result. So that is going to involve a generalized force, which is a second ranked tensor. And a thing that might happen could be a vector. It could be another second ranked tensor.

And this is going to introduce us to the nasty world of higher ranked tensor properties and their representation surfaces. So this is a nice place to quit and pause for a quiz. When we resume, we'll look at stress and strain in terms of our tensor algebra.