18.01 Exam 4

Problem 1(25 points) A solid is formed by revolving about the x-axis the region bounded by the x-axis, the line $x = 0$, the line $x = a$, and the curve,

$$y = b \sin\left(\frac{\pi x}{a}\right).$$

Find the volume of the solid.

You may use the half-angle formulas,

$$\begin{cases} \cos^2(\theta/2) & = & (1 + \cos(\theta))/2, \\ \sin^2(\theta/2) & = & (1 - \cos(\theta))/2 \end{cases}$$

Solution to Problem 1 Using the disk method, the volume is,

$$V = \int dV = \int_{x=0}^{x=a} A(x)dx$$

where $A(x)$ is the area of a cross-section disk through x,

$$A(x) = \pi y(x)^2 = \pi b^2 \sin^2(\pi x/a).$$

To integrate, use the half-angle formulas to rewrite,

$$\sin^2(\pi x/a) = (1 - \cos(2\pi x/a))/2.$$

Substituting this in gives,

$$V = \int_{x=0}^{x=a} \pi b^2 \left(\frac{1}{2} - \frac{1}{2}\cos(2\pi x/a)\right) dx.$$

Using the Fundamental Theorem of Calculus, this becomes,

$$V = \frac{\pi b^2}{2}\left(x - \frac{a}{2\pi}\sin(2\pi x/a)\right|_0^a = \frac{\pi b^2}{2}((a) - (0)).$$

Therefore, the total volume of the solid of revolution is,

$$V = \boxed{\pi a b^2/2.}$$

Problem 2(25 points) A solid is formed by revolving about the y-axis the region bounded by the x-axis, the line $x = 0$, the line $x = a$, and the curve,

$$y = \frac{ab}{x} - b.$$

Find the volume of the solid.

It is simplest to use the shell method. But you may use the disk method if you prefer.

Solution to Problem 2 Using the shell method, the volume is,

$$V = \int dV = \int_{x=0}^{x=a} A(x)dx,$$

where $A(x)$ is the area of the cylinder through x,

$$A(x) = \text{Circumference } \times \text{ Height} = (2\pi x)(y(x)) = 2\pi x(ab/x - b) = 2\pi b(a - x).$$

Thus, the total volume is,

$$V = 2\pi b \int_{x=0}^{x=a} a - xdx = 2\pi b \left(ax - \frac{x^2}{2} \right|_0^a.$$

Evaluating, the total volume is,

$$V = \boxed{\pi a^2 b.}$$

Problem 3(25 points) A surface is formed by revolving about the x-axis the curve,

$$y = x^3, \quad 0 \le x \le 1.$$

Since the curve is revolved about the x-axis, the radius of each slice is y. Compute the surface area of the surface.

Solution to Problem 3 The area of the surface of revolution is,

$$A = \int dA = \int 2\pi r ds.$$

Since the curve is revolved about the x-axis, the radius of each slice is y, $r = y$. Thus the area is,

$$A = \int_{x=0}^{x=1} 2\pi x^3 ds.$$

The differential element of arc length satisfies the "formal Pythagorean theorem",

$$(ds)^2 = (dx)^2 + (dy)^2.$$

Since the independent variable is x, express dy as,

$$dy = \frac{dy}{dx}dx = 3x^2dx.$$

Substituting in gives,

$$(ds)^2 = (dx)^2 + (3x^2dx)^2 = (1+9x^4)(dx)^2.$$

Taking square roots gives,

$$ds = \sqrt{1+9x^4}dx.$$

Substituting into the integral gives,

$$A = \int_{x=0}^{x=1} 2\pi x^3\sqrt{1+9x^4}dx.$$

To evaluate this integral, substitute,

$$\begin{array}{r|l} u = 1+9x^4 & u(1) = 10 \\ du = 36x^3dx & u(0) = 1 \end{array}$$

The integral becomes,

$$A = \int_{u=1}^{u=10} 2\pi u^{1/2}(du/36) = \frac{\pi}{18}\int_1^{10} u^{1/2}du.$$

Evaluating the integral gives,

$$A = \frac{\pi}{18}\left(\frac{2}{3}u^{3/2}\right|_1^{10}.$$

Simplifying, this becomes,

$$A = \boxed{\pi(10\sqrt{10}-1)/27.}$$

Problem 4(25 points) Sketch the polar curve,

$$r(\theta) = \sin(\theta)\sin(\theta + (\pi/2)), \quad 0 \leq \theta \leq \pi.$$

Take note: the angle θ varies over only 1/2 of a complete revolution.

In particular, label the following on your graph,

(i) in which quadrant or quadrants the curve is contained,

(ii) the endpoints of the curve,

(iii) the two slopes of the tangent lines at the endpoints of the curve,

(iv) and the angle or angles θ at which $r(\theta)$ is a maximum.

Solution to Problem 4 An image file of the polar curve is on the course webpage. The curve is contained in the first and fourth quadrants (where $x \geq 0$, but y can be positive or negative). The endpoints of the curve are both the point $(0,0)$. The slopes of the tangent lines are both 0; the tangent lines are both just the x-axis. The angles at which $r(\theta)$ is maximum are the points where $r'(\theta)$ is 0,

$$r'(\theta) = \cos(\theta)\sin(\theta + (\pi/2)) + \sin(\theta)\cos(\theta + (\pi/2)) = 0.$$

Solving gives,

$$\frac{\cos(\theta)\sin(\theta + (\pi/2))}{\sin(\theta)\cos(\theta + (\pi/2))} = -1.$$

Graphing sine and cosine, or thinking about the unit circle leads to,

$$\sin(\theta + (\pi/2)) = \cos(\theta), \quad \cos(\theta + (\pi/2)) = -\sin(\theta).$$

Thus the equation is,

$$\frac{\cos^2(\theta)}{-\sin^2(\theta)} = -1 \Leftrightarrow \tan^2(\theta) = +1.$$

In other words, $\tan(\theta)$ equals $+1$ or -1. Thus θ is 45 degrees or 135 degrees, i.e.,

$$\theta = \boxed{\pi/4, \text{ or } 3\pi/4.}$$