18.01 PRACTICE FINAL, FALL 2003

Problem 1 Find the following definite integral using integration by parts.

$$\int_0^{\frac{\pi}{2}} x\sin(x)dx.$$

Solution Let $u = x$, $dv = \sin(x)dx$. The $du = dx$, $v = -\cos(x)$. So $\int udv = uv - \int vdu$, i.e.,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\sin(x)dx = (-x\cos(x)|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos(x)dx = \\ (-\tfrac{\pi}{2}\cdot 0 + 0\cdot 1) + (\sin(x)|_0^{\frac{\pi}{2}} = 0 + (1-0) = 1.\end{aligned}$$

Problem 2 Find the following antiderivative using integration by parts.

$$\int x\sin^{-1}(x)dx.$$

Solution First substitute $x = \sin(\theta)$, $dx = \cos(\theta)d\theta$. Then the integral becomes,

$$\int \theta\sin(\theta)\cos(\theta)d\theta.$$

Of course $\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$. Thus we need to compute,

$$\int \frac{1}{2}\theta\sin(2\theta)d\theta.$$

Set $u = \frac{1}{2}\theta$, $dv = \sin(2\theta)d\theta$. Then $du = \frac{1}{2}d\theta$ and $v = -\frac{1}{2}\cos(2\theta)$. So $\int udv = uv - \int vdu$, i.e.,

$$\begin{aligned}\int \tfrac{1}{2}\theta\sin(2\theta)d\theta = -\tfrac{1}{4}\theta\cos(2\theta) + \tfrac{1}{4}\int\cos(2\theta)d\theta = \\ -\tfrac{1}{4}\theta\cos(2\theta) + \tfrac{1}{8}\sin(2\theta) + C.\end{aligned}$$

Using trigonometric formulas, this equals,

$$-\frac{1}{4}\theta(1 - 2\sin^2(\theta)) + \frac{1}{4}\sin(\theta)\sqrt{1-\sin^2(\theta)} + C.$$

Back-substituting, $\sin(\theta) = x$, gives the final answer,

$$-\frac{1}{4}(1-2x^2)\sin^{-1}(x) + \frac{1}{4}x\sqrt{1-x^2} + C.$$

Problem 3 Use L'Hospital's rule to compute the following limits.

(a) $\lim_{x\to 0}\frac{a^x - b^x}{x}$, $0 < a < b$.

(b) $\lim_{x\to 1}\frac{4x^3 - 5x + 1}{\ln x}$.

Solution (a). As x approaches 0, both the numerator and denominator approach 0. The corresponding derivatives are,

$$\frac{d}{dx}(a^x - b^x) = \ln(a)a^x - \ln(b)b^x, \quad \frac{d}{dx}(x) = 1.$$

Therefore, by L'Hospital's rule,

$$\lim_{x\to 0}\frac{a^x - b^x}{x} = \lim_{x\to 0}\frac{\ln(a)a^x - \ln(b)b^x}{1} = \ln(a) - \ln(b).$$

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(b). As x approaches 1, the numerator approaches $4-5+1=0$, and the denominator approaches $\ln(1)=0$. The corresponding derivatives are,

$$\frac{d}{dx}(4x^3-5x+1)=12x^2-5, \quad \frac{d}{dx}\ln(x)=\frac{1}{x}.$$

Therefore, by L'Hospital's rule,

$$\lim_{x\to 1}\frac{4x^3-5x+1}{\ln(x)}=\lim_{x\to 1}\frac{12x^2-5}{\frac{1}{x}}=\frac{12-5}{1}=7.$$

Problem 4 Determine whether the following improper integral converges or diverges.

$$\int_1^\infty e^{-x^2}dx.$$

(Hint: Compare with another function.)

Solution Because the integrand is nonnegative, the integral converges if and only if it is bounded. Therefore the comparison test applies. For $x>1$, $x^2>x$. Therefore $-x^2<-x$ and $0\le e^{-x^2}<e^{-x}$. Integrating,

$$\int_1^\infty e^{-x}dx=\left(-e^{-x}\right|_1^\infty=(0+e^{-1})=e^{-1}<\infty.$$

Therefore, also $\int_1^\infty e^{-x^2}dx$ converges (and is bounded above by e^{-1}).

Problem 5 You wish to design a trash can that consists of a base that is a disk of radius r, cylindrical walls of height h and radius r, and the top consists of a hemispherical dome of radius r (there is no disk between the top of the walls and the bottom of the dome; the dome rests on the top of the walls). The surface area of the can is a fixed constant A. What ratio of h to r will give the maximum volume for the can? You may use the fact that the surface area of a hemisphere of radius r is $2\pi r^2$, and the volume of a hemisphere is $\frac{2}{3}\pi r^3$.

Solution The area of the base is πr^2. The area of the sides are $2\pi rh$. The area of the dome is $2\pi r^2$. Therefore we have the equation,

$$A=\pi r^2+2\pi rh+2\pi r^2=\pi r(3r+2h).$$

It follows that $h=\frac{A}{2\pi r}-\frac{3r}{2}$. The volume of the cylindrical portion of the can is the area of the base times the height, i.e., πr^2h. The area of the dome of the can is $\frac{2}{3}\pi r^3$. Therefore the total volume of the can is,

$$V(r)=\pi r^2\left(\frac{A}{2\pi r}-\frac{3r}{2}\right)+\frac{2}{3}\pi r^3= \frac{Ar}{2}-\frac{5\pi r^3}{6}.$$

The endpoints for r are $r=0$ and $r=\sqrt{\frac{A}{3\pi}}$. The critical points for r occur when,

$$\frac{dV}{dr}=\frac{A}{2}-\frac{5\pi r^2}{2}=0,$$

i.e., $A=5\pi r^2$. Since $A=3\pi r^2+2\pi rh$, we conclude that $2\pi rh=A-3\pi r^2=2\pi r^2$. Cancelling, we have that $h=r$. This is contained in the interval for r, moreover geometric reasoning (or the first derivative test) shows this is a maximum for V. Therefore the maximum volume is obtained when $h=r$.

Problem 6 A point on the unit circle in the xy-plane moves counterclockwise at a fixed rate of $1\frac{\text{radian}}{\text{second}}$. At the moment when the angle of the point is $\theta=\frac{\pi}{4}$, what is the rate of change of the distance from the particle to the y-axis?

Solution The coordinates of the point are $(\cos(\theta), \sin(\theta))$. The distance from the y-axis is the absolute value of the x-coordinate. Since the point is in the 1st quadrant, this is just $x = \cos(\theta)$. Therefore the rate of change of the distance is,

$$\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt} = -\sin(\theta) \cdot 1 \frac{\text{radian}}{\text{second}} = -\frac{1}{\sqrt{2}} \frac{\text{radian}}{\text{second}}.$$

Problem 7 Compute the following integral using a trigonometric substitution. Don't forget to back-substitute.

$$\int \frac{x^2}{\sqrt{1-x^2}} dx.$$

Hint: Recall the half-angle formulas, $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$, $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$.

Solution This integral calls for a trigonometric substitution, $x = \sin(\theta)$, $dx = \cos(\theta)d\theta$. The integral becomes,

$$\int \frac{\sin^2(\theta)}{\cos(\theta)} \cos(\theta)d\theta = \int \sin^2(\theta)d\theta.$$

By the half-angle formulas, this is,

$$\int \frac{1}{2}(1 - \cos(2\theta))d\theta = \frac{1}{2}(\theta - \frac{1}{2}\sin(2\theta)) + C.$$

Using the double-angle formula, $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, and back-substituting $\sin(\theta) = x$ yields,

$$\frac{1}{2}\sin^{-1}(x) - \frac{1}{2}x\sqrt{1-x^2} + C.$$

Problem 8 Compute the volume of the solid of revolution obtained by rotating about the x-axis the region in the 1st quadrant of the xy-plane bounded by the axes and the curve $x^4 + r^2y^2 = r^4$.

Solution The curve intersects the y-axis when $r^2y^2 = r^4$, i.e. $y = r$. The curve intersects the x-axis when $x^4 = r^4$, i.e. $x = r$. So the endpoints of the curve are $(0, r)$ and $(r, 0)$. Using the disk method, the volume of the solid is,

$$\int_{x=0}^{x=r} \pi y^2 dx.$$

Since $y^2 = r^2 - \frac{x^4}{r^2}$, the volume is,

$$V = \int_{x=0}^{x=r} \pi r^2 - \frac{\pi x^4}{r^2} dx = \left(\pi r^2 x - \frac{\pi x^5}{5r^2} \right|_0^r.$$

This evaluates to $\pi r^3 - \frac{1}{5}\pi r^3 = \frac{4}{5}\pi r^3$.

Problem 9 Compute the area of the surface of revolution obtained by rotating about the y-axis the portion of the lemniscate $r^2 = 2a^2\cos(2\theta)$ in the 1st quadrant, i.e., $0 \leq \theta \leq \frac{\pi}{4}$.

Solution The polar equation for arclength is $ds^2 = dr^2 + r^2d\theta^2$, which is equivalent to $r^2ds^2 = r^2dr^2 + r^4d\theta^2$. By implicit differentiation,

$$2rdr = -4a^2\sin(2\theta)d\theta, \quad r^2dr^2 = 4a^4\sin^2(2\theta)d\theta^2.$$

Therefore,

$$r^2ds^2 = r^2dr^2 + r^4d\theta^2 = 4a^4\sin^2(2\theta)d\theta^2 + 4a^4\cos^2(2\theta)d\theta^2 = 4a^4d\theta^2.$$

So $ds = \frac{2a^2}{r}d\theta$.

The area of the surface of revolution is given by,

$$\int 2\pi x ds = \int_{\theta=0}^{\theta=\frac{\pi}{4}} 2\pi r\cos(\theta)\frac{2a^2}{r}d\theta = \int_0^{\frac{\pi}{4}} 4\pi a^2\cos(\theta)d\theta = \left(4\pi a^2\sin(\theta)\right|_0^{\frac{\pi}{4}}.$$

Therefore the surface area is $\frac{4\pi}{\sqrt{2}}a^2$.

Problem 10 Compute the area of the *lune* that is the region in the 1^{st} and 3^{rd} quadrants contained inside the circle with polar equation $r = 2a\cos(\theta)$ and outside the circle with polar equation $r = a$.

Solution Setting $2a\cos(\theta)$ equal to a, the points of intersection occur when $2\cos(\theta) = 1$, i.e. $\theta = -\frac{\pi}{3}$ and $\theta = +\frac{\pi}{3}$. So the lune is the region between the two graphs for $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$. The outer curve is $r_o = 2a\cos(\theta)$ and the inner curve is $r_i = a$. The formula for the area between two polar curves is

$$\text{Area} = \int \frac{1}{2}(r_o^2 - r_i^2)d\theta.$$

In this case, the area is,

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}(4a^2\cos^2(\theta) - a^2)d\theta = a^2 \int_0^{\frac{\pi}{3}} (4\cos^2(\theta) - 1)d\theta.$$

To evaluate this, use the half-angle formula, $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$. The integral becomes,

$$a^2 \int_0^{\frac{\pi}{3}} (2 + 2\cos(2\theta) - 1)d\theta = a^2 \left(\theta + \sin(2\theta)\right|_0^{\frac{\pi}{3}}.$$

Therefore the area is $a^2(\frac{\pi}{3} + \frac{\sqrt{3}}{2})$, i.e. $\frac{2\pi+3\sqrt{3}}{6}a^2$.

Problem 11 Find the equation of every tangent line to the hyperbola C with equation $y^2 - x^2 = 1$, that contains the point $(0, \frac{1}{2})$.

Solution By implicit differentiation,

$$2y\frac{dy}{dx} - 2x = 0, \quad \frac{dy}{dx} = \frac{x}{y}.$$

Therefore, the slope of the tangent line to C at (x_0, y_0) is $\frac{x_0}{y_0}$. So the equation of the tangent line to C at (x_0, y_0) is,

$$(y - y_0) = \frac{x_0}{y_0}(x - x_0).$$

If the tangent line contains the point $(0, \frac{1}{2})$, then (x_0, y_0) satisfies the equation,

$$(\frac{1}{2} - y_0) = \frac{x_0}{y_0}(0 - x_0), \quad \frac{y_0}{2} - y_0^2 = -x_0^2.$$

Of course also $y_0^2 - x_0^2 = 1$, therefore $\frac{y_0}{2} = y_0^2 - x_0^2 = 1$. So $y_0 = 2$. The two solutions of x_0 are $x_0 = \sqrt{3}$ and $x_0 = -\sqrt{3}$. The equations of the corresponding tangent lines are,

$$\begin{cases} (y-2) &= \frac{\sqrt{3}}{2}(x - \sqrt{3}), \\ (y-2) &= -\frac{\sqrt{3}}{2}(x + \sqrt{3}). \end{cases}$$

Problem 12 Compute each of the following integrals.

(a) $\int \sec^3(\theta)\tan(\theta)d\theta$.
(b) $\int \frac{x-1}{x(x+1)^2}dx$.
(c) $\int \frac{2x-1}{2x^2-2x+3}dx$.
(d) $\int \sqrt{e^{3x}}dx$.

Solution (a). Substituting $u = \sec(\theta)$, $du = \sec(\theta)\tan(\theta)d\theta$, the integral becomes,

$$\int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3(\theta) + C.$$

(b). This is a proper rational function. Use a partial fractions expansion,

$$\frac{x-1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

By the Heaviside cover-up method, $A = -1$ and $C = \frac{-2}{-1} = 2$. This only leaves B to compute. Plug in $x = 1$ to get,

$$0 = \frac{-1}{1} + \frac{B}{2} + \frac{2}{2^2} = \frac{B}{2} - \frac{1}{2}, \quad B = 1.$$

So the partial fraction decomposition is,

$$\frac{x-1}{x(x+1)^2} = \frac{-1}{x} + \frac{1}{x+1} + \frac{2}{(x+1)^2}.$$

Thus the antiderivative is,

$$\int \frac{-1}{x} + \frac{1}{x+1} + \frac{2}{(x+1)^2} dx = -\ln(x) + \ln(x+1) - \frac{2}{(x+1)} + C'.$$

(c). The derivative of the denominator is $4x - 2$. This is twice the numerator. Substituting $u = 2x^2 - 2x + 3$, $du = (4x-2)dx$, the integral becomes,

$$\int \frac{1}{u}\left(\frac{1}{2}du\right) = \frac{1}{2}\ln(u) + C = \frac{1}{2}\ln(2x^2 - 2x + 3) + C.$$

(d). Of course $\sqrt{e^{3x}} = e^{\frac{3}{2}x}$. Therefore the antiderivative is,

$$\int e^{\frac{3}{2}x} dx = \frac{2}{3}e^{\frac{3}{2}x} + C.$$