Lecture 14. October 14, 2005

Homework. Problem Set 4 Part II: Problem 2.

Practice Problems. Course Reader: 3B-1, 3B-3, 3B-4, 3B-5.

1. The problem of areas. The ancient Greeks computed the areas of triangles, quadrilaterals, and many other polygons. Their basic method was *dissection*: dissecting a polygonal region exactly into smaller regions, usually triangles, having known areas. The area of the large region is the sum of the areas of the small regions. But the ancient Greeks also knew the area of a circle, which cannot be dissected exactly into finitely many polygonal regions. Their method was *exhaustion*: finding polygonal regions *approximately* equal to the original region, and computing the limit of the areas of the polygons as the approximation improves.

Example. A regular N-sided polygon inscribed in a circle of radius r has *apothem length* $a = r\cos(\pi/N)$ and *chord length* $b = 2r\sin(\pi/N)$. Thus the area of the polygon is,

$$A = N\frac{ab}{2} = Nr^2\sin(\pi/N)\cos(\pi/N) = r^2\frac{N}{2}\sin(2\pi/N) = \pi r^2\frac{\sin(2\pi/N)}{2\pi/N}.$$

As N increases, $2\pi/N$ decreases to 0. Because $\lim_{t\to 0}\sin(t)/t$ equals 1, as N approaches infinity, the area of the polygon approaches,

$$\lim_{N\to\infty}\pi r^2\frac{\sin(2\pi/N)}{2\pi/N} = \boxed{\pi r^2.}$$

A more sophisticated version of the method of exhaustion gives the *Riemann integral*. Here is the basic problem.

Problem (Area). Find the signed area between the graph of $y = f(x)$ and the x-axis over the interval $a \le x \le b$.

For a region above the x-axis, the *signed area* is simply the area. For a region below the x-axis, the signed area is the negative of the area. For a region partly above the x-axis and partly below the x-axis, the signed area is the sum of the signed area of the region above the x-axis and the signed area of the region below the x-axis.

2. Partitions. A *partition* of an interval $[a, b]$ is a finite decomposition of the interval as a union of non-overlapping subintervals,

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-2}, x_{n-2}] \cup [x_{n-1}, x_n].$$

Since an interval is determined by its right and left endpoints, to specify a partition of $[a, b]$, it is equivalent to give an ordered sequence of increasing numbers,

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = b.$$

The kth subinterval of the partition is the interval $[x_{k-1}, x_k]$, having length,

$$\Delta x_k = x_k - x_{k-1}.$$

A partition is *fine* if the subintervals are small, and *coarse* if the subintervals are large. It may seem the number of intervals n is a good measure of fineness: since the subintervals of a fine partition are small, the number n of subintervals must be large. However, a partition into many subintervals may include a few subintervals that are quite large. For instance, the partition

$$[0, 1] = [0, 1/2n] \cup [1/2n, 2/2n] \cup [2/2n, 3/2n] \cup \cdots \cup [(n-2)/2n, (n-1)/2n] \cup [n-1/2n, n/2n] \cup [1/2, 1],$$

has n very small intervals of length $1/2n$, but has one interval, $[1/2, 1]$, of size $1/2$. The number $1/2$ may not seem large, but as n increases, it is quite large compared to $1/2n$.

Because of such examples, a better measure of fineness is mesh size: The *mesh size* of a partition is the maximal length of any subinterval in the partition,

$$\text{mesh} = \max \Delta x_k | k = 1, \ldots, n.$$

3. Riemann sums. Let $f(x)$ be a function defined on an interval $a \leq x \leq b$. Given a partition $a = x_0 < \cdots < x_n = b$ of $[a, b]$, and given a choice, for every $k = 1, \ldots, n$, of element x_k^* in the kth subinterval, $x_{k-1} \leq x_k^* \leq x_k$, the curvilinear region bounded by $y = f(x)$ and the x-axis is approximated by a union of n vertical strips. The kth vertical strip lies above or below the interval on the x-axis, $x_{k-1} \leq x \leq x_k$, and has height $y_k^* = f(x_k^*)$. The width of the vertical strip is Δx_k, thus the signed area is,

$$\Delta A_k = y_k^* \Delta x_k.$$

The total area of the union of vertical strips is simply the sum of the areas of individual vertical strips,

$$A = \sum_{k=1}^{n} y_k^* \Delta x_k.$$

The sum above is a *Riemann sum.* It is an approximation of the signed area of the curvilinear region.

There are many choices of partition. And for each partition, there are many choices for the numbers x_k^*. However, there are some special choices. On the kth interval, the smallest value $f(x)$ takes on is denoted by,

$$y_{k,\min} = \min\{f(x)|x_{k-1} \le x \le x_{k+1}\}.$$

Similarly, the largest value $f(x)$ takes on is denote by,

$$y_{k,\max} = \max\{f(x)|x_{k-1} \le x \le x_{k+1}\}.$$

For every choice of x_k^* in the kth interval, y_k^* is trapped between these two values,

$$y_{k,\min} \le y_k^* \le y_{k,\max}.$$

Denoting,

$$\Delta A_{k,\min} = y_{k,\min}\Delta x_k, \quad \Delta A_{k,\max} = y_{k,\max}\Delta x_k,$$

the area ΔA_k is trapped between these two values,

$$\Delta A_{k,\min} \le \Delta A_k \le \Delta A_{k,\max}.$$

Denoting the sums of the areas by,

$$A_{\min} = \sum_{k=1}^n \Delta A_{k,\min} = \sum_{k=1}^n y_{k,\min}\Delta x_k,$$
$$A_{\max} = \sum_{k=1}^n \Delta A_{k,\min} = \sum_{k=1}^n y_{k,\min}\Delta x_k,$$

the Riemann sum A is trapped between the two values,

$$A_{\min} \le A \le A_{\max}.$$

Thus, if $A_{\min}$ and $A_{\max}$ are close to each other, the value of A does not depend very much on the choices of the numbers x_k^*.

4. The Riemann integral. The method of the Riemann integral is to compute both $A_{\min}$ and $A_{\max}$ for a sequence of partitions whose mesh sizes approach 0. The mesh size measures the fineness of the partition, thus also the fit of the union of vertical strips to the curvilinear region. If the two limits,

$$\lim_{\text{mesh}\to 0} A_{\min}, \quad \lim_{\text{mesh}\to 0} A_{\max},$$

are defined and equal, it is said *the Riemann integral exists*, and the common limit is called the *Riemann integral*,

$$\int_a^b f(x)dx = \lim_{\text{mesh}\to 0} A_{\min} = \lim_{\text{mesh}\to 0} A_{\max}.$$

Also, $f(x)$ is said to be *Riemann integrable on the interval* $[a,b]$. Another name for the Riemann integral is the *definite integral*.

Example. Consider the function $f(x) = x$ on the interval $0 \leq x \leq L$, for some positive number L. Form the partition with n subintervals of equal length,

$$x_0 = 0 = 0L/n, x_1 = 1L/n, x_2 = 2L/n, \ldots, x_k = kL/n, \ldots x_n = nL/n = L.$$

Every interval has length $\Delta x_k = L/n$. So the mesh size is L/n. The minimum value of $f(x)$ on the interval $x_{k-1} \leq x \leq x_k$ is $y_{k,\min} = x_{k-1} = (k-1)L/n$. The maximum value is $y_{k,\max} = x_k = kL/n$. Thus,

$$A_{\min} = \sum_{k=1}^{n} y_{k,\min} \Delta x_k = \sum_{k=1}^{n} \frac{(k-1)L}{n} \frac{L}{n} = \frac{L^2}{n^2} \sum_{k=1}^{n} (k-1),$$

and,

$$A_{\max} = \sum_{k=1}^{n} y_{k,\max} \Delta x_k = \sum_{k=1}^{n} \frac{kL}{n} \frac{L}{n} = \frac{L^2}{n^2} \sum_{k=1}^{n} k.$$

To evaluate these sums, use the well-known formula,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

This also gives,

$$\sum_{k=1}^{n} (k-1) = \sum_{l=0}^{n-1} l = \sum_{l=1}^{n-1} l = \frac{(n-1)n}{2},$$

by making the substitution $l = k - 1$. Substituting the formula gives,

$$A_{\min} = \frac{L^2}{n^2} \frac{n(n-1)}{2} = \frac{L^2}{2} (1 - \frac{1}{n}),$$

and,

$$A_{\min} = \frac{L^2}{n^2} \frac{n(n+1)}{2} = \frac{L^2}{2} (1 + \frac{1}{n}).$$

Therefore,

$$\lim_{n \to \infty} A_{\min} = \frac{L^2}{2} \lim_{n \to 0} (1 - \frac{1}{n}) = \frac{L^2}{2} (1 - 0) = \frac{L^2}{2}.$$

Similarly,

$$\lim_{n \to \infty} A_{\max} = \frac{L^2}{2} \lim_{n \to 0} (1 + \frac{1}{n}) = \frac{L^2}{2} (1 + 0) = \frac{L^2}{2}.$$

Since the two limits are equal, $f(x) = x$ is Riemann integrable on the interval $[0, L]$, and,

$$\int_0^L x dx = \boxed{L^2/2.}$$

This agrees with the familiar result from high-school geometry: the area of a triangle equals one half of the base times the height, since both the base and height of this triangle are L.

5. Rules for Riemann integrals. There are several rules for Riemann integrals, summarized below.

$$
\begin{aligned}
\int_a^b (f(x) + g(x))dx &= \int_a^b f(x)dx + \int_a^b g(x)dx, \\
\int_a^b (r \ f(x))dx &= r \ \int_a^b f(x)dx, \\
\int_a^b f(x)dx + \int_b^c f(x)dx &= \int_a^c f(x)dx.
\end{aligned}
$$