Lecture 19. October 28, 2005

Homework. Problem Set 5 Part I: (d) and (e); Part II: Problems 2 and 3.

Practice Problems. Course Reader: 4A-1, 4A-3, 4B-1, 4B-3, 4B-6.

1. Differentials revisited. In a typical applied integration problem, the main difficulty is finding the integrand and the limits of integration. An unknown quantity, for instance area A, depends on some other quantity, for instance the x-coordinate. The method is to allow the independent variable x vary "infinitesimally" from x to $x + dx$ and then use geometry or science to deduce the corresponding variation dA of the unknown quantity. The deduction is usually intuitive rather than rigorous. What is important is whether the deduction leads to the correct solution. If so, the method of Riemann sums usually gives a rigorous proof of the intuitive answer. But if the solution is incorrect, no argument will prove it correct.

2. Areas between curves. Given an interval $a \leq x \leq b$ and two functions $f(x) \geq g(x)$ defined

on the interval, what is the area of the region bounded by the lines $x = a, x = b$ and the curves

$y = f(x)$, $y = g(x)$? This problem can be solved directly: the area is the difference of the area between $y = f(x)$ and the x-axis and the area between $y = g(x)$ and the x-axis. Each of these is a Riemann integral.

The differential method asks, what is the infinitesimal change in the area A as x varies from x to $x + dx$? The infinitesimal region is a rectangle of base dx and height $f(x) - g(x)$. Thus the infinitesimal change in A is,

$$dA = \text{height} \times \text{base} = (f(x) - g(x))dx.$$

Integrating gives,

$$A = \int dA = \int_{x=a}^{x=b} f(x) - g(x)dx.$$

Of course this is the same answer as in the last paragraph. But for other applied integral problems, the differential method may be the only method that works.

Example. Find the area bounded by the curve $y = x(x^2 - 3)$ and a horizontal tangent line.

The horizontal tangent lines are the tangent lines to the curve at critical points. Setting the derivative equal to 0 gives,

$$\frac{dy}{dx} = 3x^2 - 3 = 3(x-1)(x+1).$$

Thus the critical points are $x = \pm 1$. The function is odd, so symmetry suggests the area is the same regardless of the choice of critical point. Thus, choose the critical point $x = -1$. The corresponding value of the function is,

$$y = (-1)((-1)^2 - 3) = (-1)(-2) = 2.$$

This is the equation of the horizontal tangent line. Each intersection point $(b, f(b))$ of the tangent line with $y = x(x^2 - 3)$ occurs at a solution $x = b$ of,

$$x(x^2 - 3) = 2.$$

By hypothesis, $x = -1$ is a solution. Thus the polynomial factors as,

$$x^3 - 3x - 2 = (x+1)(x^2 - x - 2) = (x+1)^2(x-2).$$

The remaining intersection point is $(2, 2)$. So the area bounded by the curve $y = x(x^2 - 3)$ and the tangent line $y = 2$ is,

$$\int_{x=-1}^{x=2} 2 - (x(x^2 - 3))dx = \int_{-1}^{2} -x^3 + 3x + 2dx.$$

Using the Fundamental Theorem of Calculus, this equals,

$$\left(\frac{-x^4}{4} + \frac{3x^2}{2} + 2x \right) \Big|_{-1}^{2} = \boxed{27/4.}$$

3. Volumes of solids of revolutions: the disk method. If the region in the xy-plane bounded by $x = a$, $x = b$, $y = f(x)$ and the x-axis is revolved through xyz-space about the x-axis, what is the volume of the resulting solid? The solid is called a *solid of revolution*. The *disk method* applies the method of differentials to solve this problem. As x increases from x to $x+dx$, the corresponding infinitesimal region of the solid is essentially a disk. The width of the disk is dx. The area of the disk is $\pi[f(x)]^2$. Thus the infinitesimal volume of the disk is,

$$dV = \text{Area} \times \text{width} = \pi[f(x)]^2 dx.$$

Thus the volume is,

$$V = \int dV = \int_{x=a}^{x=b} \pi[f(x)]^2 dx.$$

Example. Find the volume of a right circular cone whose base has radius R and whose vertex has height H above the base.

The cone is the solid of revolution for the plane region bounded by $x = 0$, the x-axis, and the line containing $(0, R)$ and $(H, 0)$. The equation of this line is,

$$y = \frac{(H - x)R}{H}.$$

Thus the area of an infinitesimal disk is,

$$dV = \text{Area} \times \text{width} = \pi \frac{(H - x)^2 R^2}{H^2} dx,$$

and the volume is,

$$V = \int dV = \int_{x=0}^{x=H} \pi \frac{(H - x)^2 R^2}{H^2} dx.$$

Making the substitution $u = H - x$, $du = -dx$ gives,

$$V = \int_{u=H}^{u=0} \pi \frac{R^2}{H^2} u^2 (-du) = \pi \frac{R^2}{H^2} \left(-\frac{u^3}{3} \Big|_H^0 \right).$$

Evaluating gives the volume,

$$V = \boxed{\pi R^2 H / 3.}$$

In particular, the antiderivative of u^2 is responsible for the denominator 3 in the formula for the volume.

Example. Find the volume of a sphere of radius R.

The sphere of radius R is the solid of revolution for the plane region bounded by the x-axis and the upper semicircle $y = \sqrt{R^2 - x^2}$. Thus the volume is,

$$V = \int_{x=-R}^{x=R} \pi[\sqrt{R^2 - x^2}]^2 dx = \int_{-R}^{R} \pi(R^2 - x^2) dx = \pi \left(R^2 x - \frac{x^3}{3} \Big|_{-R}^{R} \right).$$

Evaluating gives the volume,

$$V = \boxed{4\pi R^3/3.}$$

4. The slice method. A generalization of the disk method is the *slice method*. The problem is to find the volume of a region bounded by the planes $x = a$ and $x = b$ given the knowledge of the area $A(x)$ of the *slice* of the solid by the plane containing $(x, 0, 0)$ parallel to the yz-plane. As x increases from x to $x + dx$, the corresponding infinitesimal region of the solid is essentially a cylinder. The width of the cylinder is dx. And the area is the area $A(x)$ of the slice. Thus the infinitesimal volume of the cylinder is,

$$dV = \text{Area} \times \text{width} = A(x)dx.$$

Thus the volume is,

$$V = \int dV = \int_{x=a}^{x=b} A(x)dx.$$

Example. Find the volume of the "corner" region bounded by the xy-plane, the xz-plane, the yz-plane, and the plane containing $(L, 0, 0)$, $(0, L, 0)$ and $(0, 0, L)$.

This region is bounded by the planes $x = 0$ and $x = L$. The x-slice of the region is a right isosceles triangle. The base and altitude of the triangle both equal $f(x)$, where $y = f(x)$ is the equation of the line passing through $(0, L)$ and $(L, 0)$. This equation is,

$$f(x) = L - x.$$

Thus the slice area is

$$A(x) = \frac{1}{2}\text{base} \times \text{altitude} = \frac{1}{2}(L - x)^2.$$

Thus the infinitesimal volume is,

$$dV = A(x)dx = \frac{1}{2}(L - x)^2 dx,$$

giving the total volume,

$$V = \int dV = \int_{x=0}^{x=L} \frac{1}{2}(L - x)^2 dx.$$

Make the substitution $u = L - x$, $du = -dx$ to get,

$$V = \int_{u=L}^{u=0} \frac{1}{2}u^2(-du) = \frac{1}{2}\left(-\frac{u^3}{3}\right)\Big|_0^L.$$

Evaluating gives,

$$V = \boxed{L^3/6.}$$

Thus the "corner" takes up one sixth of the corresponding cube of edge length L.

5. Volumes of solids of revolution: the washer method. A variation on the disk method is the *washer method*. A *washer* is the solid obtained by removing from a larger disk a concentric smaller disk of the same width. If the outer radius of the washer is r_o and the inner radius is r_i, then the net area of the washer is,

$$A = \pi r_o^2 - \pi r_i^2 = \pi(r_o^2 - r_i^2).$$

Thus the volume of the washer is,

$$dV = \text{Area} \times \text{width} = \pi(r_o^2 - r_i^2)dx,$$

giving a total volume,

$$V = \int dV = \int_{x=a}^{x=b} \pi(r_o^2 - r_i^2)dx.$$

Example. The main part of a dog dish is a solid of revolution whose radial cross-section is a triangle of height H whose base has inner radius R_i and outer radius R_o. Find the volume of material used to make the dog dish.

Note. This integral was only set-up in lecture. The derivation will be completed in recitation. Here is the complete derivation. Denote by x the height along the altitude of the triangle. Thus x varies from $x = 0$ to $x = H$. When $x = H$, the inner radius and outer radius are each equal to the average $(R_i + R_o)/2$ of the two radii. Both radii depend linearly on x.

The equation for the inner radius increases linearly from $r_i = R_i$ at $x = 0$ to $r_i = (R_i + R_o)/2$ at $x = H$. Thus,

$$r_i(x) = R_i \frac{H - x}{H} + \frac{R_i + R_o}{2} \frac{x}{H}.$$

Similarly, the equation for the outer radius decreases linearly from $r_o = R_o$ at $x = 0$ to $r_i = (R_i + R_o)/2$ at $x = H$. Thus,

$$r_o(x) = R_o \frac{H - x}{H} + \frac{R_i + R_o}{2} \frac{x}{H}.$$

Since $r_o^2 - r_i^2$ is a difference of squares, it equals,

$$r_o^2 - r_i^2 = (r_o + r_i)(r_o - r_i).$$

This is interesting in its own right. Using this factorization, the net area of the region between the 2 circles, called an *annulus*, equals

$$\pi(r_o^2 - r_i^2) = \left(2\pi \frac{r_o + r_i}{2}\right)(r_o - r_i).$$

Note the first factor is the circumference of the center of the annulus. And the second factor is the radial width of the annulus. Thus the area of an annulus is the circumference of the center times the radial width.

Back to the problem, the center of each washer is the same,

$$\frac{r_i + r_o}{2} = \frac{R_i + R_o}{2}.$$

And the radial width of each washer is,

$$r_o - r_i = (R_o - R_i)\frac{H - x}{H}.$$

Both of these make sense: The centers are constant because the radial cross-section is an isosceles triangle. And the width decreases linearly from $R_o - R_i$ at $x = 0$ to 0 at $x = H$. Thus the cross-section area at height x is,

$$A(x) = 2\pi\frac{R_i + R_o}{2}(R_o - R_i)\frac{H - x}{H}.$$

Thus the infinitesimal volume is,

$$dV = \text{Area} \times \text{width} = \pi(R_o^2 - R_i^2)\frac{H - x}{H}dx,$$

giving a total area,

$$V = \int dV = \int_{x=0}^{x=H} \pi(R_o^2 - R_i^2)\frac{H - x}{H}dx.$$

Substituting $u = H - x$, $du = -dx$ gives,

$$V = \int_{u=0}^{u=H} \pi(R_o^2 - R_i^2)\frac{u}{H}dx = \frac{\pi(R_o^2 - R_i^2)}{2H}\left(u^2\big|_0^H\right).$$

Thus the total volume of material to produce the dog dish is,

$$V = \boxed{\pi(R_o^2 - R_i^2)H/2.}$$

One reality check is that this is the same volume as a cylinder with the same center $(R_i + R_o)/2$ and height H as the dish, and whose (constant) radial width equals the average radial width of the dish, $(R_o - R_i)/2$.