18.01 Single Variable Calculus
Fall 2006

18.01 Practice Questions for Exam 2 Solutions. Fall 2006

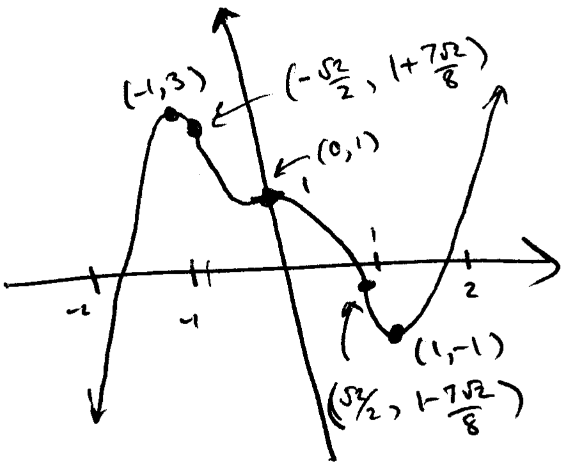
1) $f(x) = 3x^5 - 5x^3 + 1$ $\quad f'(x) = 0 \quad x = 0, \pm 1$

$f'(x) = 15x^4 - 15x^2$ $\quad f''(x) = 0 \quad x = 0, \pm \sqrt{2}/2$

$f''(x) = 60x^3 - 30x$ $\quad f(x) \to -\infty \quad x \to -\infty$

$f(x) \to +\infty \quad x \to +\infty$

x	$f(x)$	$f'(x)$	$f''(x)$	
-2	$-55<0$	\	\	\
-1	$3>0$	0	-30	loc. max.
$-\sqrt{2}/2$	$1+\frac{7\sqrt{2}}{8}$	/	0	inflection
0	1	0	0	inflection
$\sqrt{2}/2$	$1-\frac{7\sqrt{2}}{8}$	\	0	inflection
1	-1	0	30	loc. min.
2	$57>0$	\	\	\



There is an x, $-2<x<-1$, since $f(-2)<0$ and $f(-1)>0$.
There is an x, $1<x<2$, since $f(1)<0$, $f(2)>0$.
There is an x, $0<x<1$, since $f(0)>0$, $f(1)<0$.

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2) $f(x) = 4x^2 - \frac{1}{x}$ $\quad f(x) = 0, \quad x = 1/\sqrt[3]{4}$

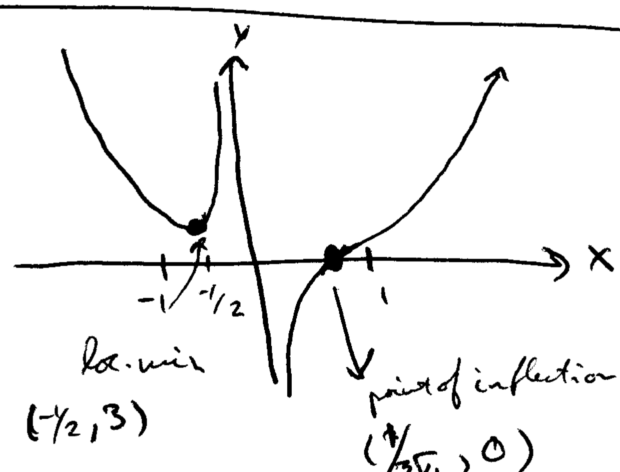
$f'(x) = 8x + 1/x^2$ $\quad f'(x) = 0, \quad x = -1/2$

$f''(x) = 8 - 2/x^3$ $\quad f''(x) = 0, \quad x = 1/\sqrt[3]{4}$

as $x \to \infty$, $f(x) \to \infty$ $\quad$ asymptote at $x = 0$

as $x \to -\infty$, $f(x) \to \infty$

$f''(-1/2) > 0$



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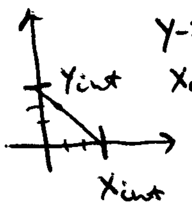
3)



$y - 2 = m(x-1)$

$X_{int} = -2/m + 1$

$Y_{int} = 2 - m$

$A = \frac{1}{2} X_{int} \cdot Y_{int} = 2 - \frac{2}{m} - m/2$ $\qquad \frac{dA}{dm} = -1/2 + 2/m^2$

$-\infty < m < 0$

$\frac{dA}{dm} = 0$ at $m = -2$

$\left.\frac{d^2A}{dm^2}\right|_{m=-2} = \left.-4/m^3\right|_{m=-2} = 1/2 > 0$ so $m=-2$, $A = 4$ is a local min.

$A \to \infty$ as $m \to 0$ or as $m \to -\infty$. So $m = -2$, $A = 4$ is the global min.

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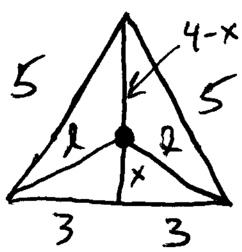
4)



$L = 2\ell + 4 - x = 2\sqrt{9+x^2} + 4 - x$ $\qquad L(\sqrt{3}) \approx 9.2 < 10$

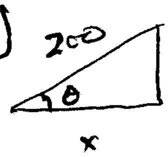
$0 \le x \le 4$ $\qquad \frac{dL}{dx} = \frac{2x}{\sqrt{9+x^2}} - 1$

$L(0) = 10$

$L(4) = 10$ $\qquad \frac{dL}{dx} = 0$ at $x = \sqrt{3}$

$L(\sqrt{3}) = 3\sqrt{3} + 4 \approx 9.2$

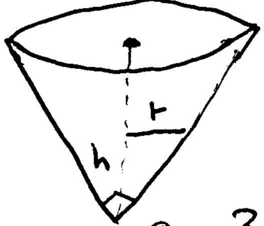
Since L at the endpoints is larger than at the unique interior crit. pt., this unique crit. pt. is a min.

5) 

$\cos\theta = \frac{x}{200}$

$-\sin\theta \frac{d\theta}{dt} = \frac{1}{200}\frac{dx}{dt}$

$\left.\frac{dx}{dt}\right|_{\theta=\pi/6} = 50 \quad \left.\frac{d\theta}{dt}\right|_{\theta=\pi/6} = \frac{1}{200}\cdot 50\cdot(-2) = -\frac{1}{2}\ \frac{\text{rad}}{\text{sec}}$

6) 

$r=h$ since this is a right circular cone.

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3$

(a) $3 = \frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$, $\left.\frac{dh}{dt}\right|_{h=2} = \frac{3}{4\pi}$

(b) Assume the rate of evaporation is proportional to the surface area, i.e. $\frac{dV}{dt} = C\pi r^2 = C\pi h^2$ (C is some negative constant).

$C\pi h^2 = \frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$ So $\boxed{\frac{dh}{dt} = \text{Const}}$

7) $f(x) = e^{-\lambda x}(1+2\sin x)^{-1}$

$\approx (1-\lambda x)(1+2x)^{-1}$

$\approx (1-\lambda x)(1-2x)$

$= 1-(2+\lambda)x + 2\lambda x^2$

So f is const. to first order if $\boxed{\lambda = -2}$

To estimate $f(x) = e^{2x}(1+2\sin x)^{-1}$ we use 2nd order approx.

$f(x) = e^{2x}(1+2\sin x)^{-1}$

$\approx \left(1+2x+\frac{(2x)^2}{2}\right)(1+2x)^{-1}$

$\approx (1+2x+2x^2)(1-2x+2x^2)$

$= 1+2x^2+\dots$

So $f(.1) \approx 1+2(.1)^2 = \boxed{1.02}$

8) From M.V.T.: for any $a<b$ there is a c $a<c<b$ s.t.

(a) $f(b)-f(a) = f'(c)(b-a) > 0$ since $f'(c)>0$, $b-a>0$.

so $f(b) > f(a)$, i.e. f is increasing.

(b) For $x>0$ $\frac{e^x - e^0}{x-0} = \left.\frac{d}{dx}(e^x)\right|_{x=c} = e^c$ for $0<c<x$

or $e^x = 1+e^c x$. For $0<c$, $e^c>1$,

so $e^x = 1+e^c x > 1+x$.

9) a) $\int \frac{dx}{(3x+2)^2} = \frac{1}{3}\int\frac{du}{u^2} = -\frac{1}{3}\frac{1}{u}+C$ $\quad u=3x+2,\ du=3dx$

$= -\frac{1}{3}\frac{1}{3x+2}+C$

b) $\int \sin(2x)\sin x\,dx = \int 2\sin^2 x\cos x\,dx$ $\quad u=\sin x,\ du=\cos x\,dx$

$= 2\int u^2 du = \frac{2}{3}u^3 + C = \frac{2}{3}\sin^3 x + C$

c) $\int\frac{(\ln x)^2}{x}dx = \int u^2 du = \frac{1}{3}u^3 + C$ $\quad u=\ln x,\ du = dx/x$

$= \frac{1}{3}(\ln x)^3 + C$

10) $\frac{dy}{dx} = x(y^2+1)$ $\quad \frac{dy}{y^2+1} = x\,dx$ $\quad \tan^{-1}y = \frac{x^2}{2}+C$

$y = \tan\left(\frac{x^2}{2}+C\right)$ $\quad 1 = \tan C,\ C = \pi/4$

$y = \tan\left(x^2/2 + \pi/4\right)$.

11) $\frac{dT}{dt} = k(T-T_e)$ $\quad \frac{dT}{T-T_e} = k\,dt$ $\quad \ln(T-T_e) = kt + C$

$T = T_e + Ae^{kt}$, (let $A = e^C$)

$20 = T(0) = 100 + A$ so $A = -80$

$30 = T(5) = 100 - 80e^{5k}$ $\quad k = \frac{1}{5}\ln 7/8$

$60 = T(t_{done}) = 100 - 80e^{kt_{done}}$

$t_{done} = \frac{5\ln 1/2}{\ln 7/8} \approx 26$