18.01 Single Variable Calculus
Fall 2006

18.01 Practice Exam 4 Solutions

1 $\displaystyle\int_0^{1/2}\frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6}\frac{\sin^2 u}{\cancel{\cos u}}\cdot\cancel{\cos u}\,du$

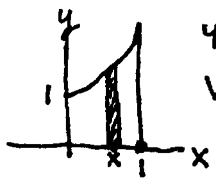
Put $x = \sin u$ $\quad \frac{1}{2} = \sin \pi/6$

(or $x = \cos u$)

$$= \int_0^{\pi/6}\frac{1-\cos 2u}{2}\,du$$

$$= \frac{u}{2} - \frac{\sin 2u}{4}\Big]_0^{\pi/6} = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

2



$y = e^x$

$$\text{Volume} = \int_0^1 2\pi x e^x\,dx$$

$$\int x e^x\,dx = xe^x - \int e^x\,dx$$
$$= xe^x - e^x$$

$$\therefore \text{volume} = 2\pi(xe^x - e^x)\Big|_0^1$$
$$= 2\pi(0-(-1)) = 2\pi$$

3 $\displaystyle\frac{4x}{(x^2-1)(x-1)} = \frac{4x}{(x-1)^2(x+1)}$

$$= \frac{2}{(x-1)^2} + \frac{B=1}{x-1} + \frac{-1}{x+1}$$

by coverup $\qquad$ by coverup

Put $x=0$: $\quad 0 = 2 - B - 1$; $\quad B = 1$

Integrating:

$$\int\frac{4x\,dx}{(x^2-1)(x+1)} = \frac{-2}{x-1} + \ln(x-1) - \ln(x+1) + c$$

$$\underset{\text{OR}}{=} \frac{-2}{x-1} + \ln\left(\frac{x-1}{x+1}\right) + c$$

4 $\displaystyle\int_a^b\sqrt{1+y'^2}\,dx$ $\qquad y = \sin^2 x$

$y' = 2\sin x\cos x$
$= \sin 2x$

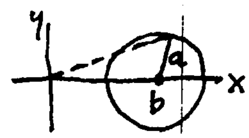
Arclength

$$= \int_0^{\pi/4}\sqrt{1+\sin^2 2x}\,dx$$

Since $0 \le \sin^2 2x \le 1$ on the interval,

$$\int_0^{\pi/4}\sqrt{1+\sin^2 2x}\,dx \le \underbrace{\pi/4}_{\text{length of interval}}\cdot\sqrt{2} < \frac{3.2}{4}\cdot\underset{\substack{\|\\ 1.2}}{\frac{3}{2}}$$

5

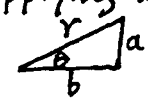


a) $(x-b)^2 + y^2 = a^2$ $\quad$ (by translation of $x^2+y^2=a^2$ to $(b,0)$ center)

$x^2 + y^2 - 2bx + b^2 = a^2$

$$\boxed{r^2 - 2br\cos\theta = a^2 - b^2}$$

b) Applying law of cosines to



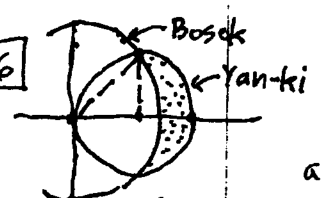
we get

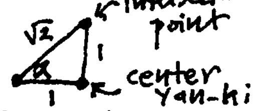
$a^2 = r^2 + b^2 - 2br\cos\theta$,

same as part (a).

DON'T HAVE TO DO!

6





shows this is a right Δ, $\alpha = \pi/4$

(or you can solve $\begin{cases} r = 2\cos\theta \\ r = \sqrt{2}\end{cases}$ simultaneously, for θ)

Using symmetry:

$$\text{Area} = 2\cdot\frac{1}{2}\int_0^{\pi/4}(2\cos\theta)^2 - (\sqrt{2})^2\,d\theta$$

b) $$= 2\int_0^{\pi/4}(\underset{\substack{\|\\ \cos 2x}}{2\cos^2\theta - 1})\,d\theta = 2\cdot\frac{\sin 2x}{2}\Big]_0^{\pi/4}$$
$$= 1$$

[By elem. geometry:  $= \frac{1}{2} + \frac{\pi}{4}$

 $= \frac{\pi(\sqrt{2})^2}{8} + A \quad \therefore A = 1/2,\ 2A = 1$]