Derivative of the Square Root Function

a) Use implicit differentiation to find the derivative of the inverse of $f(x) = x^2$ for $x > 0$.

b) Check your work by finding the inverse explicitly and then taking its derivative.

Solution

If you're having trouble with this problem, it may help to review Professor Jerison's example of the derivative of the arctangent function.

a) Use implicit differentiation to find the derivative of the inverse of $f(x) = x^2$ for $x > 0$.

We wish to find $y' = \frac{dy}{dx}$ where $y = f^{-1}(x)$ and $f(x) = x^2$. Our goal is to practice using implicit differentiation, so instead of finding $f^{-1}(x)$ right away we will follow the steps outlined in lecture.

We start with our statement $y = f^{-1}(x)$ and apply the function f to both sides:

$$\begin{aligned} y &= f^{-1}(x) \\ f(y) &= f(f^{-1}(x)) \\ y^2 &= x \end{aligned}$$

Next, we differentiate both sides of the equation then solve for y'.

$$\begin{aligned} \frac{d}{dx}y^2 &= \frac{d}{dx}x \\ 2yy' &= 1 \\ y' &= \frac{1}{2y} \end{aligned}$$

This gives us an expression describing y' in terms of y. We return to our equation $y^2 = x$ to rewrite it in terms of x. Remembering that $x > 0$, we find:

$$y = \sqrt{x}.$$

Plugging this in to our equation for y' we get:

$$y' = \frac{1}{2y} = \frac{1}{2\sqrt{x}}.$$

We conclude that if $f(x) = x^2$, the derivative of f^{-1} is $\frac{1}{2\sqrt{x}}$.

b) Check your work by finding the inverse explicitly and then taking its derivative.

If $f(x) = x^2$ then $f^{-1}(x) = \sqrt{x}$ for $x > 0$.

$$\begin{aligned} \frac{d}{dx}\sqrt{x} &= \frac{d}{dx}x^{1/2} \\ &= \frac{1}{2}x^{-1/2} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$