Linear Approximation of a Complicated Exponential

Example 3: Find the linear approximation of $f(x) = \dfrac{e^{-3x}}{\sqrt{1+x}}$ near $x = 0$. We could calculate $f'(x)$ and find $f'(0)$. But instead, we will do this by algebraically combining the linear approximations we already have.

From our list of linear approximations we have:

$$e^x \approx 1 + x$$
$$(1+x)^r \approx 1 + rx$$

So $\dfrac{1}{\sqrt{(1+x)}} = (1+x)^{-1/2} \approx 1 - \frac{x}{2}$.

We rewrite $\dfrac{e^{-3x}}{\sqrt{1+x}}$ as $e^{-3x} \cdot (1+x)^{-1/2}$ so that all we'll have to do is multiply two linear equations to get our approximation.

$$e^{-3x} \approx 1 + (-3x)$$

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} \approx 1 - \frac{1}{2}x$$

Put these two approximations together to get:

$$\frac{e^{-3x}}{\sqrt{1+x}} \approx (1 - 3x)(1 - \frac{1}{2}x)$$
$$\approx 1 - \frac{1}{2}x - 3x + \frac{3}{2}x^2$$
$$\approx 1 - \frac{1}{2}x - \frac{6}{2}x + \frac{3}{2}x^2$$

Remember that x is a number close to 0, so x^2 is a number very close to 0. In fact, when working with linear approximations we assume that x^2 is so small that we can safely ignore it. Our approximation then becomes:

$$\frac{e^{-3x}}{\sqrt{1+x}} \approx 1 - \frac{7}{2}x$$

When you're allowed to ignore all the higher degree terms — x^2, x^3 and so on — this sort of algebra becomes very simple. Why are we allowed to do that here and not in our algebra classes? For one thing, we're not calculating exact values only approximations. We know that the graph of the tangent line probably isn't the same as the graph of the function; we've already decided that finding an exact value is too hard. For another thing, these approximations are only valid when x is close to 0. If x is 1/100 then x^2 is 1/10000. The $\frac{3}{2}x^2$ we discarded has a value of 0.00015 when $x = .01$. When x is "small enough", any terms involving x^2 are so small that they don't significantly affect the final estimate.

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