PROFESSOR: Welcome back to recitation.

Today what we're going to do is use what we know about first and second derivatives and what we know about functions from way back in algebra and precalculus, to sketch a curve. So I want you to sketch the curve y equals x over 1 plus x squared. Doesn't have to be perfect, but try and use what you know about these derivatives, first and second derivatives of this function, and what you've talked about in the lecture to get a pretty good sketch of this. I'll give you a little time to work on it and then I'll be back and I'll work on it for you.

Welcome back.

So hopefully you feel good about the sketch you've drawn. But just to check everything, we can go through it together. And what I'm going to do, just to keep track of things, is I'm going to put an axis in this region and then I'm going to do all my work sort of off to the side and come back slowly. So we'll try and keep track of everything that way.

So before I do anything else I'm just going to draw myself a nice axis here. And I'll give myself even a little bit of-- oops, that's maybe a little off, but-- so we'll assume every hash mark is one unit. I'll just put a 1 there so we know every hash mark here is going to represent one unit. And I won't write the rest of them.

Now one of the things you always do first, is you want to make sure that you understand where the function is defined. So we have to check right away, are there any values of x for which this function is not defined? Well, how can that happen? If it were a logarithm or if it were a square root function we would have problems in the domain where would have to check and make sure that the input was positive. In this case, because we have a rational function, we have to make sure that the denominator is never equal to 0.

But if you notice, the denominator is 1 plus x squared. Well, x squared is always bigger than or equal to 0, and once I add 1, I'm in the clear. I'm always positive in the denominator. So the denominator is always positive, so I don't have to put any vertical asymptotes.

Some other things we think about before we even start taking derivatives, or anything I can find out about this function, like end behavior. When we say end behavior we mean, what happens as x goes to positive infinity and as x goes to negative infinity? And from what you've seen before, as x goes to positive infinity, because this is a rational function, the higher power is going to win out. The higher power always wins out. So the higher power here is in the denominator, so as x goes to positive infinity this whole expression is going to head to 0. For large values of x the x squared is significantly bigger than the x. And so the denominator is significantly bigger than the numerator. That's how we can think about this. So when x goes to plus or minus infinity we know that our function is going to be headed to 0, so it has a horizontal asymptote.

OK. And then another thing we would-- we should notice is the sign of the graph. Notice where the sign will change. This denominator is always positive so the sign of the function depends completely on the numerator. And so when the numerator is positive this function will be positive. When the numerator is negative this function will be negative.

So that's a little bit that we should keep in mind. And now let's go to using our derivatives to figure out a little bit more.

So obviously, first I should take some derivatives and then we'll look at what we can get out of them. So let's let f of x equal x over 1 plus x squared. So then f prime of x, what do we get? We get 1 plus x squared minus x times 2x over 1 plus x squared, squared. So I'm just going to continue that straight below. Let's see. I can keep this x squared minus 2x squared, gives me a 1 minus x squared, in the numerator, over 1 plus x squared, quantity squared.

OK. I'm going to keep that right here. We're going to do a little bit of calculation below in a moment, but I'm going to record the second derivative just to the right. So the second derivative, remember, is the derivative of the first derivative. So now I'm going to take this derivative, again using the quotient rule, which I used here.

So the derivative of the top is minus 2x and then times 1 plus x squared squared and then I subtract the derivative of the bottom times the top. So I'll keep the top here, 1 minus x squared. And then the derivative of the bottom has a little chain rule on it, so I'm going to get a times 2 times 1 plus x squared times 2x. And then this whole thing is over 1 plus-- whoa-- x plus 1. We'll write x squared plus 1 to the fourth. Sorry to switch the direction or the order of those.

OK. Now I'm going to pull out a 1 plus x squared from the numerator to simplify it. And then I'm going to see what I have left. Here I have a 1 plus x squared times a negative 2x. That's going to be negative 2x minus 2 x cubed. Here I'm going to have-- 2 times 2 is 4x times this 1 minus x squared. So I have a minus 4x plus 4 x squared-- cubed, sorry. Let's make sure. So I should have a 4x here and then an x squared times 4x, which is 4 x cubed. And that sign should be positive. And then I still have to divide by 1 plus x squared to the fourth.

To make this much simpler I'm just going to divide out one of the 1 plus x squareds, simplify what's inside, and we'll leave it that way. Actually, let me move this down so there's a little more room. So the numerator will now be 2 x cubed minus 6x over 1 plus x squared to the third.

So these were some tools that we needed. Now we're going to try and use them.

So let's recall what we know. We know that when the derivative is equal to 0, we have a maximum or minimum for the function. And we know that when the second derivative is equal to 0, we have changes in concavity. So let's find those places. Let's find where the first derivative is 0 and let's find where the second derivative is 0. So I'm going to work under each individual function to do that.

So where is f prime equal to 0? Well, f prime is only equal to 0 when the numerator is equal to 0. So let's solve 1 minus x squared equals 0. Well that's-- there's a couple ways you can think about that. You could factor it and then solve, or you could see right away this is going to be x is plus or minus 1. You get the same thing if you factor. But we see x is equal to plus or minus 1. So those are our maximum values or minimum values for the function. OK. So we know that this is an important spot for the x-value and that's an important spot for the x-value.

Now let's just come over here and look at, when is the second derivative equal to 0? So the second derivative is equal to 0, again, when the numerator is equal to 0. So let's look at what we get. Well, if we factor that we get 2x times x squared minus 3 equals 0. So this has three places it's going to be equal to 0. It's going to be equal to 0 at 0, x equals 0, and it's going to be equal to 0 at plus or minus root 3, which is sort of unfortunate that we don't know exactly where that is, but we know it's between 1 and 2. I think it's about 1.7 or something like this.

So we know we're interested in the point x equals 0 and the points x equal plus or minus square root of 3. So these are our places of interest. And so let's evaluate at least a couple of these places and see what's going on. Let's go back to the graph to do this.

Now I want to point out something I didn't say earlier, which is, if you know the function is defined everywhere, what you might want to do is evaluate the function at x equals 0 right away. It's an easy place to evaluate it. It gives you sort of a launching point.

So if I evaluate this at x equals 0 I get 0. So I know the point (0, 0) is on the graph. So I know that's one point. And now what I'm interested in, if you think about-- we know where maxes or mins occur, we know a max or min occurs at x equals plus or minus 1. Or we have a hope for a max or min there. It's a critical point, at least.

So I can evaluate the function-- sorry-- I can evaluate the function at 1 and at negative 1 and I can then plot those points. So when x is 1, I get 1 over 1 plus 1 squared, so I get 1/2. So with input 1 I get output 1/2. I'm going to erase that 1 now so we don't lose track of what's happening. That looks potentially like it could be a maximum, given sort of what's happening here, to the left.

So let's plug in negative 1 for x. I get a negative 1 over 1 plus quantity negative 1 squared. So I get negative 1 over 2, so I get negative 1/2. So at x equals negative 1, I get negative 1/2.

And let's recall what we know about the end behavior, which we said at the beginning. The end behavior of this is as x goes to positive infinity, the function's outputs go to 0. Which tells you that, in fact, this has to be a maximum. There are the only two places where the function can change direction from going up to going down, or from going down to going up. So it has to be that this is a maximum. It has to be that this is a minimum.

So, and also notice 0, based on what we know about the second derivative, is one of the inflection points. So that's also representing a place where the derivative is changing sign. So maybe the derivative was increasing and then it's going to start decreasing.

So let's look-- I think I might have said something a little off there, so I'm going to maybe come back and see if I have to fix anything in a moment-- but let me draw a rough sketch of what's happening. Very rough, very roughly we know we're going up and then we're going down. We're going down here and then we have to go back up because the end behavior.

So we have three inflection points-- this is what I want to point out-- we have three inflection points. We have an inflection point at 0 and at plus or minus root 3. So we said root 3 is bigger than 1, it's less than 2. So I know somewhere in here I have an inflection point, which represents a change in the concavity. Right? Which represents how the derivative is going to change the direction, whether it's continuing to get more negative and then getting more positive than it was previously. So yeah, that's where-- we're looking at where the derivative changes sign. As I said before.

So let me point out-- this is a change in concavity. Maybe right about in this x region we want to change concavity, and then this x region we want to change concavity. So the graph will look something like going up, going down, going down. And then I've tried to represent the change in concavity changing that direction there.

And I'm doing something that I didn't tell you yet. But if you notice, this looks highly symmetric, doesn't it? And in fact, one thing I didn't tell you about this function-- that maybe you picked up on already-- is that when I take the right-hand side and I rotate it about the origin I get the left hand side. Why is that? That's because this is an odd function. Why is it an odd function? Because the numerator is an odd function and the denominator is an even function. And so the quotient is an odd function.

So this is, I would say, a fairly good sketch of the curve y equals x over 1 plus x squared. So hopefully yours looked something like this.

And that's where we'll stop.