Ring on a String

We're going to do one more max/min problem.

Consider a ring on a string held fixed at two ends at $(0, 0)$ and (a, b) (see Fig. 1). The ring is free to slide to any point. Find the position (x, y) that the ring slides to.

Note that if $b = 0$, i.e. if the two ends are at equal heights, the ring will settle midway between the two ends ($x = \frac{a}{2}$). We can perform this experiment physically and see the result; we now want to explain that result mathematically. One reason to be interested in this problem is that it's one of many problems that must be solved in order to build a suspension bridge.

Professor Jerison drew a diagram of the possible positions of the ring in lecture by tracing the position of an actual ring on a string held by two students. The next step after drawing this diagram is to name and label the variables, as shown in Figure 1.

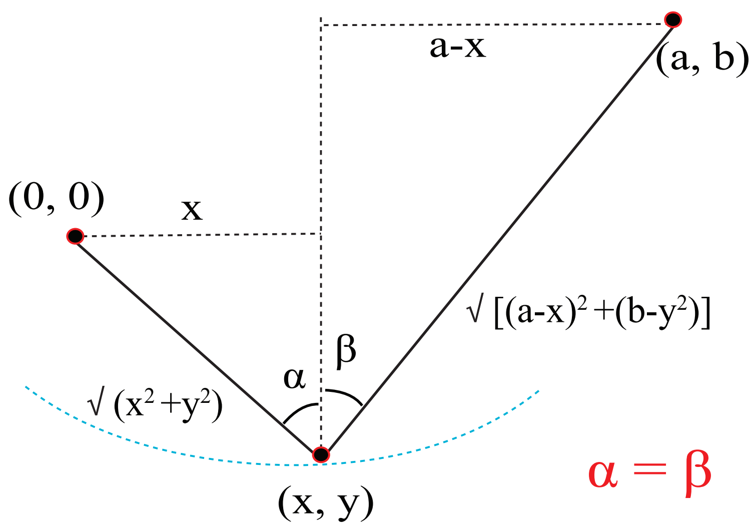

Figure 1: Illustration of the Ring on a String problem.

Physical Principle: The ring settles at the lowest height (lowest potential energy), so the problem is to minimize y subject to the constraint that (x, y) is on the string.

Constraint: The length L of the string is fixed.

$$\sqrt{x^2 + y^2} + \sqrt{(x-a)^2 + (y-b)^2} = L.$$

The function $y = y(x)$ is determined implicitly by the constraint equation above. We traced the constraint curve (possible positions of the ring) on the blackboard; the curve is also suggested in blue in Figure 1. This curve is an ellipse with foci at $(0,0)$ and (a,b), but knowing that the curve is an ellipse does not help us find the lowest point.

Experiments with the hanging ring show that the lowest point is somewhere between $x = 0$ and $x = a$. (This is one way we can confirm that the minimum solution isn't at one of the ends of the string; don't try to use the second derivative test.) Since the ends of the constraint curve are higher than the middle, the lowest point is a critical point (a point where $y'(x) = 0$). In class we also gave a physical demonstration of this by drawing the horizontal tangent at the lowest point.

To find the critical point, differentiate the constraint equation implicitly with respect to x:

$$\frac{x + yy'}{\sqrt{x^2 + y^2}} + \frac{x - a + (y - b)y'}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

Since $y' = 0$ a the critical point, the equation can be rewritten as:

$$\frac{x}{\sqrt{x^2 + y^2}} = \frac{a - x}{\sqrt{(x-a)^2 + (y-b)^2}}$$

From Fig. 1, we see that the last equation can be interpreted geometrically as saying that:

$$\sin\alpha = \sin\beta \quad \Longrightarrow \quad \alpha = \beta,$$

where α and β are the angles the left and right portions of the string make with the vertical.

Physical and geometric conclusions

The angles α and β are equal.

Using vectors to compute the force exerted by gravity on the two halves of the string, one finds that there is *equal tension* in the two halves of the string — a physical equilibrium. This is desirable in construction; if one end is under more stress than the other, it's more likely to break.

From another point of view, the equal angle property expresses a geometric property of ellipses: Suppose that the ellipse is a mirror. A ray of light from the focus $(0,0)$ reflects off the mirror according to the rule angle of incidence equals angle of reflection, and therefore the ray goes directly to the other focus at (a,b). This was used to good effect in the "Strokes of Genius: Mini Golf by Artists" exhibit at the DeCordova museum in the early 1990's; by placing the tee at one focus of an ellipse and the hole at the other, an artist created a golf course on which any stroke would end with a hole in one.

Formulae for x and y

We did not yet find the location of (x,y). We will now show that:

$$x = \frac{a}{2}\left(1 - \frac{b}{\sqrt{L^2 - a^2}}\right), \quad y = \frac{1}{2}\left(b - \sqrt{L^2 - a^2}\right).$$

Because $\alpha = \beta$,

$$x = \sqrt{x^2 + y^2}\ \sin\alpha; \quad a - x = \sqrt{(x-a)^2 + (y-b)^2}\ \sin\alpha$$

Adding these two equations,

$$a = \left(\sqrt{x^2+y^2} + \sqrt{(x-a)^2+(y-b)^2}\right)\sin\alpha = L\sin\alpha \implies \sin\alpha = \frac{a}{L}$$

The equations for the vertical legs of the right triangles are (note that $y < 0$):

$$-y = \sqrt{x^2 + y^2}\ \cos\alpha; \quad b - y = \sqrt{(x-a)^2 + (y-b)^2}\ \cos\beta.$$

Adding these two equations, and using $\alpha = \beta$, we get:

$$b - 2y = \left(\sqrt{x^2+y^2} + \sqrt{(x-a)^2+(y-b)^2}\right)\cos\alpha = L\cos\alpha$$

$$\implies$$

$$y = \frac{1}{2}(b - L\cos\alpha).$$

Use the relation $\sin\alpha = \dfrac{a}{L}$ to write:

$$\begin{aligned} L\cos\alpha &= L\sqrt{1-\sin^2\alpha} \\ &= \sqrt{L^2 - a^2}. \end{aligned}$$

Then the formula for y is:

$$y = \frac{1}{2}\left(b - \sqrt{L^2 - a^2}\right).$$

Finally, to find the formula for x, use similar right triangles:

$$\tan\alpha = \frac{x}{-y} = \frac{a-x}{b-y} \implies x(b-y) = (-y)(a-x) \implies (b-2y)x = -ay$$

Therefore,

$$x = \frac{-ay}{b-2y} = \frac{a}{2}\left(1 - \frac{b}{\sqrt{L^2-a^2}}\right).$$

Thus we have formulae for x and y in terms of a, b and L.

This derivation of the formulae for x and y wasn't covered in lecture because it is long and because the most illuminating part of the problem is the balance condition $\alpha = \beta$ that is an immediate consequence of the critical point computation.

Final Remark. In 18.02, you will learn to treat constrained max/min problems in any number of variables using a method called Lagrange multipliers.