Cube Root of x

Show that for any non-zero starting point x_0, Newton's method will never find the exact value x for which $x^3 = 0$.

Solution

Given a value x_0 close to some x where $f(x) = 0$, Newton's method usually produces a series of values x_0, x_1, x_2, ... whose values approach x. However, there is no guarantee that any of those values x_i actually equal x.

Newton's method generates a sequence according to the formula:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

In our example, $f(x) = x^3$ and $f'(x) = 3x^2$, so:

$$\begin{aligned} x_{k+1} &= x_k - \frac{x_k^3}{3x_k^2} \\ &= x_k - \frac{x_k}{3} \\ x_{k+1} &= \frac{2}{3}x_k \end{aligned}$$

Starting with x_0, Newton's method generates the sequence:

$$\begin{aligned} x_1 &= \frac{2}{3}x_0 \\ x_2 &= \frac{2}{3}x_1 = \frac{2}{3} \cdot \left(\frac{2}{3}x_0\right) = \frac{2^2}{3^2}x_0 \\ x_3 &= \frac{2}{3}x_2 = \frac{2^3}{3^3}x_0 \\ x_4 &= \frac{2}{3}x_3 = \frac{2^4}{3^4}x_0 \\ &\vdots \\ x_n &= \frac{2^n}{3^n}x_0 \end{aligned}$$

If $x_0 \neq 0$, the values of the x_i get closer and closer to the desired value 0 but never exactly equal zero.