Generalizing the Mean Value Theorem – Taylor's theorem

We explore generalizations of the Mean Value Theorem, which lead to error estimates for Taylor polynomials. Then we test this generalization on polynomial functions.

Recall that the mean value theorem says that, given a continuous function f on a closed interval $[a, b]$, which is differentiable on (a, b), then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Rearranging terms, we can make this look very much like the linear approximation for $f(b)$ using the tangent line at $x = a$:

$$f(b) = f(a) + f'(c)(b - a)$$

except that the term $f'(a)$ has been *replaced* by $f'(c)$ for some point c in order to achieve an *exact* equality. Remember that the Mean Value Theorem only gives the existence of such a point c, and not a method for how to find c.

We understand this equation as saying that the difference between $f(b)$ and $f(a)$ is given by an expression resembling the next term in the Taylor polynomial. Here $f(a)$ is a "0-th degree" Taylor polynomial.

Repeating this for the first degree approximation, we might expect:

$$f(b) = [f(a) + f'(a)(b - a)] + \frac{f''(c)}{2}(b - a)^2$$

for some c in (a, b). The term in square brackets is precisely the linear approximation.

Question: Guess the formula for the difference between $f(b)$ and its n-th order Taylor polynomial at $x = a$. Test your answer using the cubic polynomial $f(x) = x^3 + 2x + 1$ using a quadratic approximation for $f(3)$ at $x = 1$.

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