

## Generalizing the Mean Value Theorem – Taylor’s theorem

We explore generalizations of the Mean Value Theorem, which lead to error estimates for Taylor polynomials. Then we test this generalization on polynomial functions.

Recall that the mean value theorem says that, given a continuous function  $f$  on a closed interval  $[a, b]$ , which is differentiable on  $(a, b)$ , then there is a number  $c$  in  $(a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Rearranging terms, we can make this look very much like the linear approximation for  $f(b)$  using the tangent line at  $x = a$ :

$$f(b) = f(a) + f'(c)(b - a)$$

except that the term  $f'(a)$  has been *replaced* by  $f'(c)$  for some point  $c$  in order to achieve an *exact* equality. Remember that the Mean Value Theorem only gives the existence of such a point  $c$ , and not a method for how to find  $c$ .

We understand this equation as saying that the difference between  $f(b)$  and  $f(a)$  is given by an expression resembling the next term in the Taylor polynomial. Here  $f(a)$  is a “0-th degree” Taylor polynomial.

Repeating this for the first degree approximation, we might expect:

$$f(b) = [f(a) + f'(a)(b - a)] + \frac{f''(c)}{2}(b - a)^2$$

for some  $c$  in  $(a, b)$ . The term in square brackets is precisely the linear approximation.

**Question:** Guess the formula for the difference between  $f(b)$  and its  $n$ -th order Taylor polynomial at  $x = a$ . Test your answer using the cubic polynomial  $f(x) = x^3 + 2x + 1$  using a quadratic approximation for  $f(3)$  at  $x = 1$ .

**Solution:**

The difference between  $f(b)$  and its  $n$ -th order Taylor polynomial at  $x = a$  follows a similar pattern. Recall that the  $n$ -th order Taylor polynomial  $P(x)$  for  $f(x)$  at  $x = a$  has the form:

$$P(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

So we might guess that

$$f(b) - P(b) = \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}$$

for some  $c$  in  $(a, b)$ . A proof of this fact can be found in many advanced calculus texts. We will not prove it here, but instead content ourselves with verifying it in a single special case described above.

The polynomial  $f(x)$  above has quadratic approximation  $P(x)$  at  $x = 1$  given by

$$P(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2 = 4 + 5(x - 1) + 3(x - 1)^2$$

Now we examine the difference between  $f(3) = 34$  and  $P(3) = 26$ . So we seek a  $c$  in  $(1, 3)$  such that

$$\frac{f^{(3)}(c)}{3!}(3-1)^3 = f(3) - P(3) = 8$$

But  $f$  is a cubic function, so  $f^{(3)}(x)$  is a constant independent of  $c$ . In fact, it is 6, canceling with the  $3!$  in the denominator, and we verify that the left-hand side above is indeed 8.

In practice, Taylor's theorem is quite useful. If we can bound the size of  $f^{(n+1)}(x)$  on the interval  $(a, b)$ , then we may apply the theorem to obtain concrete error estimates for approximating  $f(x)$  by an  $n$ -th degree Taylor polynomial.

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