Using Differentials to Study Population Dynamics

We have seen that differentials give a convenient way for expressing linear approximations. In this example, we explore population dynamics in the language of differentials.

A simple generational model of population dynamics says that an initial population x will yield a next generation with population given by a function $P(x)$. The next generation after that is given by "iterating" the function P, that is, $P(P(x))$. We can keep applying P to the result to find the population of successive generations. Note in particular that population will be stable over generations at any x such that $P(x) = x$. Such an x is known as a "fixed point."

We say that a fixed point x_0 is "attracting" if, given an initial population value $x_0 + \Delta x$ with Δx sufficiently small, the successive generations have size closer and closer to x_0. More formally, the sequence of values

$$x_0 + \Delta x, P(x_0 + \Delta x), P(P(x_0 + \Delta x)), P(P(P(x_0 + \Delta x))), \ldots$$

gets closer and closer to x_0.

Question:

- Show that if x_0 is a fixed point of $P(x)$ and $|P'(x_0)| < 1$, then x_0 is attracting.
- Given fixed positive constants a, b with $ab > 1$, find the fixed points of $P(x) = ax(b - x)$ and determine if they are attracting.