Using Differentials to Study Population Dynamics

We have seen that differentials give a convenient way for expressing linear approximations. In this example, we explore population dynamics in the language of differentials.

A simple generational model of population dynamics says that an initial population x will yield a next generation with population given by a function $P(x)$. The next generation after that is given by "iterating" the function P, that is, $P(P(x))$. We can keep applying P to the result to find the population of successive generations. Note in particular that population will be stable over generations at any x such that $P(x) = x$. Such an x is known as a "fixed point."

We say that a fixed point x_0 is "attracting" if, given an initial population value $x_0 + \Delta x$ with Δx sufficiently small, the successive generations have size closer and closer to x_0. More formally, the sequence of values

$$x_0 + \Delta x, P(x_0 + \Delta x), P(P(x_0 + \Delta x)), P(P(P(x_0 + \Delta x))), \dots$$

gets closer and closer to x_0.

Question:

- Show that if x_0 is a fixed point of $P(x)$ and $|P'(x_0)| < 1$, then x_0 is attracting.
- Given fixed positive constants a, b with $ab > 1$, find the fixed points of $P(x) = ax(b - x)$ and determine if they are attracting.

Solution:

For the first question, we consider $P(x_0 + \Delta x)$ for some small value of Δx. By linear approximation,

$$\begin{aligned} P(x_0 + \Delta x) &\approx P(x_0) + P'(x_0)\Delta x \\ &= x_0 + P'(x_0)\Delta x \end{aligned}$$

where, in the last step, we have replaced $P(x_0)$ by x_0 since it is a fixed point. Using differential notation instead, we may write

$$P(x_0 + dx) \approx x_0 + P'(x_0)dx$$

If $|P'(x_0)| < 1$, then $P'(x_0)dx$ is smaller than dx, and so $x_0 + P'(x_0)dx$ is closer than $x_0 + dx$ to x_0. If we define $dx_1 = P'(x_0)dx$, then we may repeat this process, applying P to $x_0 + dx_1$. By the same reasoning as above, the result will be closer than $x_0 + dx_1$ to x_0. (To make this precise, we'd need to be careful about the error in this approximation when dx is taken "sufficiently small.")

For the second question, we begin by finding the fixed points. To do this, we must solve $P(x) = x$. That is, we seek solutions x_0 to

$$ax(b - x) = x.$$

Simplifying, we have

$$(ab-1)x - ax^2 = 0$$

which gives the two solutions $x = 0$ and $x = \frac{ab-1}{a}$. To check if these are attracting fixed points, it remains to analyze the size of the derivative $P'(x_0)$ in each case.

$$P'(x) = ab - 2ax$$

so $P'(0) = ab > 1$ by assumption, hence not attracting. On the other hand, $P'(\frac{ab-1}{a}) = 2 - ab$, so our second fixed point is attracting provided that $1 < ab < 3$.

In practice, these constants a and b would be determined by certain constraints on a population such as scarcity of resources, presence of predators, etc. Attracting fixed points allow for a population to remain in relative equilibrium over time, despite small scale fluctuations in the population not considered by the model.

Finally, we note there's really no difference to using our earlier notation for linear approximation over the differential notation, except that perhaps the differential notation emphasizes very small changes in the in the independent variable x rather than arbitrary changes in x. Because this example focused on such small changes, we chose to use differential notation.