Welcome back to recitation. In this video, we're going to be working on establishing the best technique for finding an integral or finding an antiderivative. We'll be doing this, as you've seen probably in a lot of these videos, in a row. And so in this one in particular, we're going to work on these two.

So what I'd like us to find, is for the letter A, I'd like us to find an actual value if we take the integral from minus 1 to 0 of this fraction. 5 x squared minus 2x plus 3 over the quantity x squared plus 1 times x minus one.

And then, the second problem, we're just going to be finding an antiderivative. So it's finding an antiderivative of the function 1 over x plus 1 times the square root of negative x squared minus 2x. Now, this does have a domain over which this function is well-defined, as long as what's inside the square root is positive. And there are values of x for which that's positive. I just didn't want us to have to compute this one exactly. So we're just looking for an antiderivative here.

So the goal is, figure out what strategy you want to use, work through that strategy, and then I'll be back, and I'll show you which strategy I picked, and the solution that I got.

OK. Welcome back. Well, hopefully you were able to make some headway in both of these. And so what we'll do right away, is just we'll start with the first one. So on the first one, we should be able to get an actual numerical value at the conclusion of the problem. And if you look at the first one, it's probably pretty obvious you want to use partial fraction decomposition. You already have a denominator that's factored, and so this is going to be fairly easy to do.

Now, what's one thing you want to check with partial fractions, is you want to make sure that the degree of the numerator is smaller than the degree of the denominator. Notice that the numerator degree is 2 and the denominator degree is 3, because we have an x squared times x. So we don't have to do any long division. We can just start the problem.

So what I'm going to do, is I'm going to actually decompose it without showing you how I did it. And you've done that practice enough, so I'm just going to show you what I got with my decomposition, and we'll go from there.

So when I decompose, for letter A, I actually get two integrals. And the first one I get is the integral from minus 1 to 0 of 2x over x squared plus 1 dx. And the second one I get is the integral, minus 1 to 0, of 3 over x minus 1 dx. Let me just double check and make sure-- yes. That's what I got when I did this problem earlier. So this is not magic. I actually did this already. That's how I got these.

And now from here, we just have to determine-- we just have to integrate both of these. Now, what would be the strategy at this point? Well this one-- if this had just been a 2 and no x, you'd be dealing with an arctan type of problem. Because you'd be integrating 1 over x squared plus 1. But actually, because we have this 2x here, this is really a substitution problem. Right? The derivative of x squared plus 1 is 2x. Right? So we see that really what we're integrating is something like du over u. And so if you did this substitution problem, you should get something like natural log of x squared plus 1.

Let me just double check and make sure that gives me the derivative here. The derivative of natural log of x squared plus 1 is 1 over x squared plus 1 times 2x. That gives me exactly what's here. I don't need absolute values here, because this is always positive. And I know I need to evaluate it at minus 1 and 0. OK? So that takes care of the left one.

Now, the right one is-- again, it's a pretty straightforward one, because it's just 3 over x minus 1. That's a natural log again. So this natural log is even simpler. It's going to be 3 times the natural log absolute x minus 1 from minus 1 to 0.

And so now we just have to plug in everything. OK? So let's just do this one step at a time, starting over here. So the natural log, when I put in 0, I get the natural log of 1. That's 0. And I subtract what I get when I put in negative 1 for x. And negative 1 squared gives me 1, so this is minus the natural log of 2. And then I have plus 3 times whatever's over here.

So now let's look at this. When I plug in 0, I get natural log of 0 minus 1, absolute value. That's natural log of 1. That's zero again. And then I get a minus. And then I put in negative 1. Negative 1 minus 1, negative 2, absolute value, so it's natural log of 2. And so if I look it at all the way across, I see I have a negative natural log of 2 and then I have 3 natural logs of 2. So the final answer is just negative 4 natural log of 2. And that is where we'll stop with (a).

OK. So let me just remind you, actually, before we go to (b). What we did in (a) was we did partial fraction decomposition. And I gave you the numerators. And then on the first one, we had to use maybe a substitution to figure it out. I didn't write explicitly the substitution, but a substitution gives us that integral, and this one is directly a natural log.

OK. Now let's look at (b). So (b)-- let me rewrite the problem, because it's now a little far away. I think it's x plus 1 square root of negative x squared minus 2x. OK. So (b), the reason-- I wanted to make sure we did a trig substitution in a particular way, because I haven't demonstrated those very much. So the denominator wound up looking a little awkward, to force you to do it in that way. But what we want to do, is we want to actually complete the square on what's in here. And that might make you a little bit nervous. But let me just do a little sidebar work down here, and point out what we get. If we factor out a negative here, we get an x squared plus 2x. OK? So we're going to complete the square on the inside.

Now this might make some people nervous. They might say, you've a negative under the square root. But I want to point out that I have a negative here, but I could always make x squared plus 2x a negative number, and then I would have-- the negative times a negative is a positive. For instance, I think negative 1, right, if I put a negative 1 for x, I get negative 2 plus 1 is a negative value. So if I put a negative 1 for x, I'm taking the square root of a positive number. So there are values of x that make this under the square root positive. OK? So you don't have to worry about that.

Now, if I want to complete the square on what's in here, what do I do? I have x squared plus 2x. I obviously need to add a 1 to complete the square. Why is that? Because you take what's here, you divide it by 2, and you square it. And so this actually equals square root of negative x squared plus 2x plus 1. And I want to subtract 1 so that I haven't changed anything, but when I pull it out from the negative, it's another plus 1. OK?

Let's make sure we buy that. I've added 1 inside here, so if I add 1 on the outside, this is actually a minus 1, and so this is a plus 1, so together they add up to 0. So I haven't changed what's in the square root.

So if I come back and put that in right here, what do I get? I get the integral dx over x plus 1 square root-- let me move this over. I'm going to bring this to the front-- 1 minus x plus 1 squared.

All right. So from here, we have to do a trig substitution. Now, what trig substitution we want to do, we can do sine or cosine. But I'm going to do cosine, because I like secants better than cosecants, because I have those memorized better. So that's why I'm choosing cosine. You'll see why I chose that way in a little bit. But it would be, you will get the same answer if you do sine.

OK. So I'm going to come to the other side. Let's see. So I'm choosing cosine theta equals x plus 1. That's the substitution I'm making. And why am I making that substitution? I'm making that substitution because when I do 1 minus x plus 1 squared, that's actually 1 minus cosine squared theta. So that's, this in here is sine squared theta. And when I take the square root, I just get a sine theta.

So that should be pretty familiar to you by now, this strategy. But the point I'm making is that x plus 1 will be a cosine theta, and this whole square root is what becomes a sine theta. So you've seen that a fair amount, but just to remind you that.

And then the other thing we need is to replace the dx. So the dx is going to be, derivative of cosine is negative sine, so you're going to get negative sine theta d theta.

So now we know all the pieces. We said this was cosine, we said the square root is sine, and the dx is negative sine d theta. So let's rewrite that over here. So we have-- I'm going to put the negative in front, so I don't have to deal with it anymore. Negative sine theta over cosine theta sine theta d theta. These divide out, and I get negative 1 over cosine theta, which is just equal to negative secant theta. OK? So I have negative secant theta. Let me actually write that here. Negative integral of secant theta d theta.

And now what is that? Well, we know how to integrate secant. So let me write that in terms of theta. It's going to be negative natural log absolute value secant theta plus tangent theta. And then we have the plus c out here. What's the point of this? Well, we should maybe have this memorized. If you have to look it up, you have to look it up, but you saw this one in class. And the negative is just dropping down here, so don't think I added that negative in when I was taking antiderivative. It was already there.

All right. So we're done. Oh, but we're not done. Why are we not done? We're not done, because we started off with something in terms of x, and now we have something in terms of theta, so we have to finish up. And how we do that, is we go back, we look at the substitution we made. If we make a triangle based on that substitution, we figure out the values of secant theta and tangent theta, and then we can plug those in terms of x.

So let's remind ourselves-- I'm going to draw the triangle in the middle here. Let's remind ourselves of the relationship we had between theta and x. If this is theta, we said cosine theta, right here, cosine theta was equal to x plus 1. Cosine theta is adjacent over hypotenuse. So we want to say, this is x plus 1, and this is 1. And that implies by the Pythagorean theorem, that this is square root of 1 minus quantity x plus 1 squared. Let me move that over.

Notice, then, this also makes sense, why sine theta is what it is. Sine theta is this value divided by 1. So that also helps you understand that. All right. So now what do we need to read off? We need to read off secant, and we need to read off tangent. So secant is 1 over cosine, so actually, we could have gotten that one for free, from the cosine. So this 1 over cosine is 1 over x plus 1. So this thing is equal to negative natural log absolute value 1 over x plus 1 plus-- now, what's tangent? If I come back and look at the triangle, tangent theta is opposite over adjacent. Right? So I can actually just put it all over x plus 1 if I wanted. But I already started writing it separately, so I'll leave it like this. Square root of 1 minus x plus 1 quantity squared. And then close that, and then my plus c.

So now I'm actually finished with the problem. Because now I have an antiderivative in terms of x.

So let me just remind you where this problem, where we started the problem, kind of take us through quickly, and then we'll be done.

So back to the beginning, what we had, was we had an integral that was a fractional problem, but we had an x plus 1 here, and then we had this really messy-looking quadratic in here. To make it easy to deal with, I factored out a negative sign, and then I saw I could complete the square. Once you complete the square, you actually get another x plus 1 in there, which helps us to see immediately, it should be a trig substitution. So the substitution that's natural to make, because you have a 1 minus something involving an x, is going to be either cosine or sine. I chose cosine. If you'd chosen sine, you probably would have gotten a cosecant up there, instead of a secant, when you were taking an antiderivative at the very end. So you would have gotten the same answer because of the substitutions in the end.

But so I chose cosine theta is equal to x plus 1. You do that, you can replace this with cosine, this with a sine, this becomes a negative sine, and then you start simplifying. So once we came over here and simplified, we got it into something we recognize. We got it into secant. We know the antiderivative for secant, in terms of secant and tangent. We know it's exactly this. And then we went back to the relationship we had. We made ourselves a triangle in terms of the theta and the x-values. And then we were able to substitute in for secant and tangent.

All right. So hopefully that was successful for you. And that's where I'll stop.