Hi. Welcome back to recitation. In lecture, you've learned a whole bunch of different integration techniques. So I have here a couple of problems that will give you an opportunity to apply those techniques, and figure out which ones to apply.

So in particular I have one definite integral, the integral 0 to 2 of the quantity x times e to the 1 minus x squared with respect to x. And the second one is an indefinite integral of 2 arctan x divided by x squared with respect to x. So why don't you pause the video, take some time to work these out, come back, and we can work them out together.

Welcome back. Hopefully you had some luck working on these integrals. Let's get started. We can do the first one first, and then we'll go on to the second one. So here we want to compute the integral from 0 to 2 of x e to the 1 minus x squared dx.

So in order to do this, we have to, you know-- well, we could look at it and say, do I know the answer immediately, off the top of my head? And this is not one of the, you know, ones that we have immediately already memorized a formula for. So then you say, OK. So now I need to do something in order to try to integrate it. And so, you know, you have a whole bunch of techniques. And for this one, when you look at it, the technique that is going to work for you is just the simplest technique that you have, which is just a straightforward substitution.

And one way to figure that out, is that you can look at this and you can see that here we have a composition of functions, this e to the 1 minus x squared, and so whenever you have a composition of functions, there's the opportunity for there to have been a chain rule somewhere. And in this case, we see that this composition of functions is multiplied by x, and x is very closely related to the derivative of 1 minus x squared. So that's what's suggesting for us a nice substitution. And so the substitution we're going to try, then, is that we're going to put this inside function, this 1 minus x squared, we're going to set u equal to that. So, all right. So this is equal to-- so I'm going to say, u equal 1 minus x squared. So now if I take a differential, if u equals 1 minus x squared, that means that du is minus 2x dx.

And since this is a definite integral, I also need to change the bounds of integration. So when x is equal to 0, that means that u is equal to, well, 1 minus 0 squared is 1, so that's the lower bound. And when x is equal to 2, that means that u is equal to 1 minus 2 squared, which is negative 3. So we make these substitutions. OK. So what does our integral become? So this becomes the integral-- so now the lower bound becomes u equals 1, and the upper bound is u equals negative 3.

OK. So now e to the 1 minus x squared, that's just e to the u. And x dx-- well, that's almost du. It's du divided by

negative 2. So du divided by negative 2.

So we make the substitution. We transform our integral to this form. Now, this is a very nice, simple integral, right? This is just the integral of e to the u with a, you know, a constant multiple. So, OK. So we can do this right away. So this is-- I'll pull the minus 1/2 out in front. And now fundamental theorem of calculus. Antiderivative of the e to the u is just e to the u. And I have to take that between my bounds. So between u equals 1 and u equals minus 3.

So notice that because we changed bounds, we don't have to go back and convert everything back to x. We can just plug in directly. So OK. So we plug in directly. What do I get? So first I get minus 1/2 times-- all right, so I'll just pull the constant out in front. So the first term is e to the minus third minus e to the 1. OK. And if I wanted to, I could rewrite this as e minus e to the minus 3 over 2, or in a variety of other ways.

All right. So this one was a pretty straightforward integration by substitution. Definite integral, change bounds, and at the end you get this answer. So let's take a look at the second one. So the second one is a bit trickier, I think, and it requires a bit more work. So here it is. Let me rewrite it. So it's the antiderivative of 2 arctan x over x squared dx.

So we have here an antiderivative that we have to compute. And this is a kind of messy-looking expression that we're working with. All right? So this is not very pleasant. So there are a couple of different things we could think to do. So one thing that you could think to do is you could say, I really don't like arctan. I'm going to try and get rid of arctan. And the way I'm going to get rid of arctan is by doing a substitution, u equals arctan x, or x equals tan u.

So if you do that, that'll get rid of your inverse trigonometric function, but it'll introduce, in the bottom, you have this x squared, so that'll introduce like a tan u squared, and dx will give you some more trig functions. So what you'll be left with, then-- so this will be u on top. So you'll have u times a product of trig functions.

So I guess it's up to you whether you think that a polynomial, or a power of u times a bunch of trig functions, is simpler than an inverse trig function times a power. It's not clear to me that it's actually that much simpler. And in any case, we wouldn't sort of be finished at that stage. There would still be a fair amount more work to do.

So we can think about, what other things could we do? That's kind of the only obvious substitution here. But another thing that suggests itself, is that we could try an integration by parts. And one reason is, this is a product of things. You know, here it's arctan x times 1 over x squared. And arctan x is something that works nicely with integration by parts if you can differentiate it. Right?

So inverse trigonometric functions behave like logarithms with respect to integration by parts. By which I mean, they really like to be differentiated. Because when you differentiate an arctan, you get something much simpler. Well, simpler, anyhow. Maybe not much simpler. You get 1 over 1 plus x squared. So if you can apply integration

by parts and differentiate the arctan, that makes your life nice. And so, then, in order for that to work, you need for everything else to be something you can antidifferentiate, and here everything else is just 2 over x squared, and so that's something that we know how to integrate. It's easy to do.

So, OK. So let's give that a try, then. So we're going to do integration by parts, so I'm going to set u equal to arctan x. So that means u prime is equal to 1 over 1 plus x squared. And I'm going to let v prime equal, well, I guess I have to pick up the 2 somewhere, so I might as well pick it up here. 2 over x squared. And so then v, so I integrate 2 over x squared, so that gives me x to the minus 1, with a minus sign. So it's minus 2 over x.

So these are u, u prime, v and v prime. So now OK, so now I apply integration by parts, and that tells me this is equal to-- well, let's see. So it's u*v, which is minus 2 arctan x over x. That's u*v. Minus the integral of, now it's the second part, it's v u prime dx. So here v u prime is-- well, OK. So they're minus 2. I'm going to pull that out front. So it's plus 2-- all right. And then the rest of it is 1 over x times 1 plus x squared dx. So if you apply integration by parts, you have the u*v stuff that gets kicked out out front, and then what you're left with is this second integral.

Now this is, to me this is simpler-looking than what we started with. It's still not simple, but it's simpler. So it seems to me that, you know, having reached this stage, we can say, OK. We've made some progress, and now we have to keep going. Right?

So it's not obvious to me what the antiderivative of this expression should be. So how can I figure that out? Well, there are a couple of different options here. One thing you might look at this and see, is you might see-- again, I have a 1 plus x squared in the denominator, and so that might make you tempted to do a trig substitution again. And in fact, you could do a trig substitution here. You could put x equal tan theta, and you would be able to solve the question like this.

But I don't think it's the simplest way to go. Because in addition-- well, on the one hand, this has this 1 plus x squared in the denominator, but on the other hand, this is just a rational function. Yeah? It's a ratio of two polynomials. The polynomial on top is just 1, and the polynomial on the bottom is this product, x times 1 plus x squared. And so whenever you have a rational function that you're trying to integrate, you can also always use-- what's it called-- partial fraction decomposition.

So we can try to use partial fractions on the second expression. And so I think I'd like to do it that way. So let's do that. Let's do it down here. So partial fractions on the quantity 1 over x times 1 plus x squared.

So if you remember your partial fractions here, this is completely factored. This quadratic doesn't factor. It doesn't have any real roots, doesn't factor. So OK, so we have a single linear term, and a single non-factorable quadratic term. So partial fractions tells us that we can write this in the form A over x plus-- and now because the second

term is quadratic, it needs to be B*x plus C over 1 plus x squared. Right? So when you have a quadratic in the bottom, you get two constants up at the top, a linear polynomial. And then you can always check. You have three constants here, and the degree of the denominator is 3, so that's one way of checking that you're not completely off base, that the degree down here should always agree with the number of constants that you're solving for.

OK. Good. So now we need to solve this for A, B, and C. And so OK, because we have this nice single linear factor, we can do the cover up method there. So we cover up x, and we cover up over x, and we cover up everything else. So on the right-hand side, we just end up with A, and on the left-hand side, well, so x. So that's what we need, we get whatever when we plug in x equals 0. So on this side, we get 1 over 1 plus 0 squared, so that's just 1.

So the cover up method gives us that A is equal to 1. And now we need to solve for B and C. And probably the most straightforward way, in this case, to solve for B and C, is you can always just multiply through. And if you multiply through-- OK. So we'll multiply through on the left-hand-- we'll clear denominators, we'll multiply through by x times 1 plus x squared. So on the left we'll get just 1. On the right-- OK. So we multiply A over x times x times 1 plus x squared, so that gives us A times 1 plus x squared, but we know that A is 1, so this is-- the first term becomes 1 plus x squared, and the second term we multiply through by x times 1 plus x squared, and the 1 plus x squareds cancel, and we're left with plus x times B*x plus C.

Now, if you, maybe you could-- at this point, there are a number of different ways to finish. But one, for example, is you could see, all right, if you subtract the 1 plus x squared over to the other side, you have minus x squared, and over here, you have B x squared plus C*x. So for those things to be equal, you need B to be negative 1, and you need C to be 0. Right?

So OK. So good. So we have B equals minus 1, C equals 0. So the partial fraction decomposition, then, if we plug this in, is 1 over x minus x over 1 plus x squared.

So this is the partial fraction decomposition of this rational function. So you apply that partial fraction decomposition. What do you do? All right, So now you carry that back upstairs to the integral we were working on. So our integral that we started with is equal to, now-- well, so this constant is still out in front. So it's minus 2 arctan x over x plus-- so now we've got 2 times-- all right. So first, this is just algebra that we've done, so we can just substitute it in. So we replace this 1 over x times 1 plus x squared by 1 over x minus x over 1 plus x squared dx. OK. And now using the properties of integration, this is a difference of two-- an integral of a difference is the difference of the two integrals. So we just integrate them separately, right? So this is equal to-- so OK, the part out front never changes. 2 arctan x over x, plus 2 times-- OK. So you integrate 1 over x, and that just gives you ln of absolute value of x.

And OK, so you integrate-- so the second part is x over 1 plus x squared. Now, in the worst case, we would need to do some more work on this term and split things up. And one of them, we might have to complete the square or something. But in this case, it actually has worked out kind of nicely. This x over 1 plus x squared is a simple thing to handle. That's just ln of 1 plus x squared. Well, actually, one half of ln of 1 plus x squared. So this is minus 1/2 ln of 1 plus x squared. I could write absolute value, but 1 plus x squared is always positive, so it doesn't matter if I do or not. And if I wanted, I could rewrite this a bit. I could rewrite this as-- just to finish it, I could write 2 arctan x over x plus 2 ln absolute value of x minus ln of 1 plus x squared.

And I was doing an antiderivative. So at some point, whenever I finished the last one, I should have added a constant. So this should have had a plus c here, and it should have a plus c there. Right, good.

So OK, so there's my answer. And let's just recap quickly how we arrived at it. So we started with this integral. And we saw, we recognized it as a good candidate for integration by parts, and so we applied integration by parts, seeing that the arctan part was the part that really wanted to be differentiated, and that the 1 over x squared part, you know, it could have been differentiated, it could've been integrated, but since the arctan wanted to be differentiated, this got integrated.

Good. So we did integration by parts, and then we're left with a rational function as the piece of the integral we had left to compute. So when you have a rational function, one method that always works is that you can do partial fractions. Now, in this case, this was a fairly simple partial fractions to do. First of all, the degree of the numerator was smaller than the degree of the denominator, so we didn't have to divide by anything, you know, we didn't have to do long division. And the denominator had a fairly simple form, so we could just do the cover up method, and then multiply through and be done fairly quickly.

But in any case. So we did, we applied partial fractions, so we got this expression. So then we carried that back up to our integral, and integrated. OK, so then we had this integral with the partial fraction expression in it, and that was easy to integrate. So in this case, quite easy to integrate. Sometimes it's a little messier, but we came out pretty luckily this time. So OK. And so there you go. So that's going to be our final answer.

Now, this was long and complicated. And so sometimes, when you do a long and complicated computation, you worry, did I make any stupid mistakes along the way? One way to check, always, when you've done an antiderivative, is you could always check, you could take the derivative of this expression. So that'll require you to use some product rule and some chain rule. But, you know, it's arithmetically a little difficult, but not sort of intellectually, right? You just are applying rules sort of automatically. So you can take the derivative of this expression, and check to make sure that it actually agrees with the thing that we started with.

So good. I'll end there.