

Partial Fractions – Big Example

We've seen how to do partial fractions in several special cases; now we'll do a big example so that you can see how all these cases fit together.

Remember that partial fractions is a method for breaking up rational expressions into integrable pieces. The good news is, it always works. The bad news is that it can take a lot of time to make it work.

- **Step 0** Long Division:

$$\frac{P(x)}{Q(x)} = \text{quotient} + \frac{R(x)}{Q(x)}.$$

By completing this step you split your rational function into an easy to integrate quotient and a rational function for which the degree of the denominator is greater than the degree of the numerator.

- **Step 1** Factor the Denominator $Q(x)$.

For example, suppose our remainder term looks like:

$$\frac{R(x)}{(x+2)^4(x^2+2x+3)(x^2+4)^3}$$

where the degree of $R(x)$ is less than 12. Polynomials can be extremely difficult to factor; we may need a machine to do this. This can be the hardest step in this method.

If we expand the denominator in the example we get something like:

$$x^{12} + 10x^{11} + 55x^{10} + 224x^9 + 716x^8 + 1856x^7 + 4000x^6 + 7168x^5 + 10624x^4 + 12800x^3 + 12032x^2 + 8192x + 3072$$

Factoring this polynomial by hand would be unpleasant.

- **Step 2** Set-up:

$$\frac{R(x)}{(x+2)^4(x^2+2x+3)(x^2+4)^3} = \frac{A_1}{(x+2)} + \frac{A_2}{(x+2)^2} + \frac{A_3}{(x+2)^3} + \frac{A_4}{(x+2)^4} + \frac{B_0x + C_0}{x^2+2x+3} + \frac{B_1x + C_1}{(x^2+4)} + \frac{B_2x + C_2}{(x^2+4)^2} + \frac{B_3x + C_3}{(x^2+4)^3}$$

Note that repeated quadratic factors in the denominator are treated very much the same way as repeated linear factors. There's one term for each power of the repeated factor, and the degree of the numerator is the same in each term.

There are 12 unknowns in this equation; that's not a coincidence. The degree of the denominator is 12. The numerator $R(x)$ can have at most 12 coefficients a_0, a_1, \dots, a_{11} ; i.e. the number of degrees of freedom of a polynomial of degree 11 is 12.

This is a very complicated system of equations: twelve equations for twelve unknowns. Machines handle this very well, but human beings have a little trouble.

• **Step 3** Cover-up.

We can use the cover-up method to solve for A_4 . That reduces the problem to eleven equations in eleven unknowns.

Question: I see that there are twelve unknowns, but isn't it just one big equation?

Answer: If you multiply both sides by $Q(x)$ it becomes a polynomial equation. On one side you have the known polynomial:

$$R(x) = a_{11}x^{11} + a_{10}x^{10} + \dots$$

On the other side of the equation you'll have a polynomial whose coefficients are linear combinations of the unknowns:

$$A_1(x+2)^3(x^2+2x+3)(x^2+4)^3 + A_2(\dots$$

When you set the coefficients on both sides equal to each other you get 12 equations in 12 unknowns.

Question: Should I write down all this stuff?

Answer: That's a good question! You'll notice that Professor Jerison didn't write it down. It's pages long. You're a human being, not a machine; don't try this at home.

Remember that once we've decomposed $\frac{P(x)}{Q(x)}$ into simpler fractions we still need to integrate it. The quotient and the fractions of the form $\frac{A_i}{(x+2)^i}$ are easy to integrate. However, we'll also need to compute something like:

$$\int \frac{x}{(x^2+4)^3} dx = -\frac{1}{4}(x^2+4)^{-2} + c$$

using advanced guessing or substitution of $u = x^2 + 4$. To calculate something like:

$$\int \frac{dx}{(x^2+4)^3}$$

we'd need to use the trig substitution $x = 2 \tan u$, $dx = 2 \sec^2 u du$

$$\begin{aligned} \int \frac{dx}{(x^2+4)^3} &= \int \frac{2 \sec^2 u du}{(4 \sec^2 u)^3} \\ &= \frac{2}{64} \int \cos^4 u du \\ &= \frac{1}{32} \int \left(\frac{1 + \cos(2u)}{2} \right)^2 du \\ &\vdots \end{aligned}$$

When calculating:

$$\int \frac{dx}{x^2 + 2x + 3} = \int \frac{dx}{(x+1)^2 + 2},$$

we'll have to complete the square and then use a trig substitution to get something like:

$$\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x+2}{\sqrt{2}} \right) + c.$$

In addition, the integral of:

$$\int \frac{x}{x^2 + 2x + 3} dx$$

is an expression involving $\ln(x^2 + 2x + 3)$. In theory, we know how to do each of these twelve integrals. In practice it will take a long time.

There's no easier method. The method we use to compute these integrals is going to be at least as complicated as the results, and we've seen that the results can get very complicated. But this method always works, and there are computer programs that can do these calculations for us.

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