PROFESSOR: Welcome back to recitation. In this video I want us to work on some problems looking at integrals that may converge or diverge. So these are improper integrals. And we want to know if each of the integrals below converges or diverges. And if they converge, I want you to compute them. I want you to actually evaluate it, find a number that is the area under the curve. So that's really, remember, what the integral is. So we should be able to find, actually, the finite number that represents that.

So there are three of them. The first one is the integral from 0 to infinity of cosine x dx. The second one is the integral from 0 to 1 of natural log x divided by x to the 1/2 dx. So that's just square root x, down there. And the third one is the integral from minus 1 to 1 of x to the minus 2/3 dx. So why don't you take a while to work on this. Pause the video. When you're feeling good about your answers bring the video back up. I'll be back then to show you how I do them.

OK, welcome back. Well hopefully you were able to answer these questions. The question again was that we wanted to know if the following intervals-- intervals-- integrals converged or diverged. And then I wanted us to actually compute them if they converged. So again, we're going to start with the integral of cosine x dx. Then we'll look at the integral of natural log x over root x dx. And then we'll look at the integral from minus 1 to 1 of x to the minus 2/3 dx.

So let's look at the first one. And I'll rewrite it up here so we have it. OK. And this is kind of interesting because maybe this is a little bit different than what you have seen previously in the way of saying this is potentially improper. Because cosine x certainly doesn't blow up anywhere. It's bounded between minus 1 and 1. So the function itself is not blowing up. But let's look at what we get when we try and evaluate this.

So if we take this integral, we know the antiderivative of cosine is negative-- no it's just sine. Sorry, it's just sine x. Right? The derivative of sine is cosine. So we get sine x. And what we're supposed to do, is that we're supposed to take-- let me rewrite that x. That's horrible. We're supposed to take the limit as b goes to infinity of sine x evaluated from 0 to b. Right? And so this is pretty straightforward. Again now it's the limit at b goes to infinity of sine b, because sine of 0 is 0.

Now here's where we run into trouble, because this limit doesn't exist. Right? because as b goes to infinity, sine b, the function sine x-- so now it's really the function sine b, b is now the variable, if we think about it that way-- as b goes to infinity, sine is going to oscillate as it always does between minus 1 and 1. And it's going to continue to do that. It's not going to approach a certain value and stay arbitrarily close to that value as be goes off to infinity. So this limit does not exist. And maybe what's informative is to think about how could this happen as an integral? If we know that the integral we're looking for is really the signed area under the curve.

So let me explain briefly what's happening. Let me just draw a quick picture of cosine and explain briefly what's happening. So cosine starts off like this. Right? So what happens if I wanted to integrate cosine x from 0 to infinity. I first pick up this much area. Sorry this graph is a little sloped, I realize. I pick up this much area and that's all positive. And then as I keep moving over here to here, I pick up the same amount of area, but it's negative. So once I get to here my integral, it's 0. The area above and the area below are equal.

So then I start the process again with the value 0. And so I accumulate some negative area. Then over here it's the same amount of positive area, and it kills it off. So I start off, I have some positive value, and then it becomes less positive and goes to 0. Then I get some negative value accumulated. Then it comes up and goes to 0 again. So the point is as I'm moving off to infinity, the area is oscillating. And the area, remember, is actually what the value of sine is at that point. The area from under the curve from 0 to any value for cosine x is the value of sine at that point. That's what we're seeing here.

So the point is that this integral, even though cosine x is a bounded function, the area is accumulating, then disappearing, then becoming negative, then disappearing, then accumulating, then disappearing. So there's no value that it's approaching. It's varying between all these values over and over again. So this is a weird one, maybe, where the integral doesn't exist. But it actually, it doesn't exists there. So hopefully that makes sense, and even though it's a little different, you understand the idea.

So now I'm going to go to (b). So what is (b)? It's the integral from 0 to 1 of ln x over x to 1/2 dx. OK. And what I want to point is we probably want to see why is this improper. Well at 1, we don't of a problem because natural log of 1 is 0. And we can put a 1 down here and nothing bad happens. At 0 we have a problem. The natural log of 0 actually doesn't exist. The limit as x goes to 0 of natural log x is minus infinity. And then natural log-- or sorry, natural log? Zero, evaluating square root x at 0, gives you 0. So we have something going to minus infinity in the numerator and 0 in the denominator. And we're trying to integrate that function as it goes towards that value. So we have to figure out kind of what's going on here.

So let's not worry about the bounds at the moment. Those are obviously going to be important at the end. But let's figure out what we get as an antiderivative for this. OK? And we don't worry about the constant, remember, in the antiderivative because we're going to evaluate.

So what's the best way to attack this one? Well probably you should see that you got a natural log and you've got a power of x. So this is really set up to do an integration by parts. Because remember, you like to take derivatives of natural log. And powers of x are happy to be integrated or to take derivatives of them. The only power of x that's maybe a little bit annoying to integrate is x to the minus 1, because its integral, because its antiderivative is natural log instead of, like, another power of x. But, this one is not that case so it's looking good for an integration by parts.

So again we said for an integration by parts, natural log you love to take its derivative. So we're going to let u be natural log of x and u prime be 1 over x. And then v prime is x to the minus 1/2. Right? This is x to the 1/2 in the denominator, so v prime is actually x to the minus 1/2. So v is going to be x to the 1/2 with a correction factor. So it's going to need a 2 in front, I believe. Let me double check. Derivative of this is-- 1/2 times 2 is 1-- x to the minus 1/2. So I'm doing OK.

So now what do I get when I want to take the integral? I take u*v. So it's going to be 2 x to the 1/2 ln x-- and then now I'm going to put in the bounds-- evaluated from 0 to 1, minus the integral of v u prime. So v is x to the 1/2, and u prime is x to the minus 1, really it's 1 over x. And so I'm going to keep the 2 in front, integral from 0 to 1, x to the 1/2 over x is x to the minus 1/2. This will be nice because we've already taken an antiderivative once, so we know what we get.

All right, now this one we'll have to look at, because notice that as x goes to 0 we have to figure out if this has a limit, OK? We have to figure out of this has a limit. And we'll probably need to use L'Hopital's rule to do that.

So let's finish this part up first. So I'll just keep this here. 0 to 1, that's a 0, and then minus. OK, x to the minus 1/2, its antiderivative was 2 x to the 1/2. So I have another 2, so I get a 4 c to the 1/2 evaluated from 0 to 1.

So this is easy because here, I just get 4. Right? This part right here, when I evaluate it I'll just get 4 and the minus sign in front. So this part is just negative 4, because at x equals 1 I get 1 and at x equals 0 I get 0. So that's fine. This part is fine.

This part, when I put in 1 for x, notice what happens. I get a 1 here. Natural log of 1 is 0, and so I get 0. And so the question really is, what is the limit as x goes to 0 of this quantity? I know right here-- again, I have a minus 4 here. And then the question is what do I have here? it? It could blow up still. It could diverge. We're going to see what happens.

So let me just put a line. It's the same problem, but I want to distinguish for us. So we're actually wanting to find the limit as b goes to 0 of 2 b to the 1/2 natural log b. Right? That's what we're interested in. That's the only part we don't yet understand. So let's see what happens.

Well, if I want to make this into a L'Hopital's rule thing, right now I have 0 times negative infinity. So to put it into a form I recognize, I'm going to rewrite this. This is actually equal to the limit as b goes to 0-- I'm going to keep the natural log be up here, and I'm going to write this as b to the minus 1/2 in the denominator.

Let's make sure we understand what just happened. I had a b to the 1/2 in the numerator. If I put it to the minus 1/2 in the denominator, it's still equal to the same thing. Right? b to the minus 1/2 in the denominator is equal to b to the 1/2 in the numerator. OK? Really I'm taking 1 over this, and then I'm dividing by it. That's the way you want to think about it. You want to think about saying I'm taking 1 over this, and I'm taking it in the numerator and the denominator. so I end up with it just here. That might have been confusing. The real point is this quantity, one over this quantity is equal to that quality.

And now what's the point? Why did I bother to do that? As b goes to 0, this goes to negative infinity and this goes to infinity. So now I have something where I can apply L'Hopital's rule. So the limit as b goes to 0, derivative of natural log of b is 1 over b. So I get 2 over b in the numerator. The derivative of b to the minus 1/2 is negative 1/2 b to the minus 3/2.

There's a lot of denominators in the numerator and denominator, so let's simplify this. That's the limit as b goes to 0. OK, what do we get here? We have 2 times minus 1/2, or 2 divided by minus 1/2. So I'm going to get a negative 4 from this part, the coefficient. Then I have a b to the minus 1 up here divided by a b to the minus 3/2. That's actually going to be a b to the 3/2 divided by b. That's going to be b to the 1/2.

That's algebra. You can check it if you need to. I mean, you should have gotten this. But if you didn't get this, check again. Let me make sure I get it again, b to the 3/2 divided by b, b to the 1/2. And that equals 0, because it was a limit as b goes to 0 of this quantity. Now I've just got a continuous function. I am, notice, I I didn't actually--

I kind of cheated a little bit, because I just wrote limit as b goes to 0. But I'm always doing as b goes to 0 from the right-hand side. I'm starting at 1. So you may have seen this, this 0 plus means I'm only interested as b goes to 0 from above 0. OK? I didn't write that in, but notice our integral was between 0 and 1. So it only mattered values to the right of 0. So this function is defined to the right of 0. So I can just evaluate there. I can just plug it in. It's continuous, and so I can just say at 0 it equals 0.

So now let's go back to where we were. What were we doing here? We were taking this whole piece, was to come back in here and figure out what this value was. We knew at 1, we got 0. And now we know at 0, we also get 0. So that question mark I can replace by a 0, and I get 0 minus 4. So the actual answer is negative 4.

OK, we have one more. What's the last one? The last one-- let me write it down over here again-- is the integral from minus 1 to 1 of x to the minus 2/3 dx. Now this is interesting, because this, you actually saw an example kind of like this in the lecture that at this endpoint, it's fine. You can evaluate the function at that endpoint. At this endpoint it's fine. You can evaluate the function there. So it looks good at the endpoints. But the point is that at x equals 0, because this minus power is putting your x in the denominator, you actually, your function is blowing up. It has a vertical asymptote at x equals 0.

So what we want to do is we have this strategy for these problems, is to split it up into two parts, minus 1 to 0 of x to the minus 2/3 dx plus the integral from 0 to 1 of x to the minus 2/3 dx. Now the point is that now the only place where I have a vertical asymptote, I see it as an endpoint on both of these. And this is again, just this good thing we have an additive property for these integrals, that the sum of these two integrals is going to equal this one here. As long as, you know, as long as these are converging, then I can say that if this converges and this converges, then their sum converges to this one here. OK, so that's really what I'm trying to exploit here.

Now what is the antiderivative here for x to the minus 2/3? This is going to be-- again, I'm going to write something down and then I'm going to check it. I think it should be 5/3, x to the 5/3. And then I have to multiply by 3/5. Let's double check, because this is where I always might make mistakes. If I take the derivative of 5/3 times 3/5 it gives me a 1. 5/3 minus 1 is 5/3 minus 3/3, and that's 2/3, and that's wrong. Right?

Because I went too big. It's supposed to be 1/3. I knew I was going to make a mistake. Right? I'm supposed to add 1 to minus 2/3. That's just 1/3. And so I should have a 3 in front. So for those of you who were saying she's making a mistake, I was. OK, x to the 1/3. So I take the derivative times 3, I get a 1 there. I subtract 1 and I get minus 2/3. So now I'm in business. OK. And now I just need to evaluate that here, here if I can, then another one here from 0 to 1.

And the good news is x to the 1/3 is continuous at all these points. It's continuous across negative 1, 0, 0, and 1. So I can actually just evaluate. I don't have to worry. There is a value for the function there. It's continuous through all these points. So I can just evaluate them. So let's see what I get.

I get 3 times 0. And then I get minus 3 times negative 1, so I get a negative 3 plus 3 times 1, and then minus 3 times 0. OK, so 3 times 0 minus negative 3. So that's a 3 plus 3, and I get 6.

And what's the picture of this? Well you should think about what the picture of x to the minus 2/3 looks like. And you should notice that across 0, it's even. Right? So it's actually going to have-- it's going to look the same in the left-hand side and the right-hand side. So if I had wanted to, I could have just found the value of say, this integral-- I like positive numbers better-- the integral from 0 to 1 of this function, multiplied it by 2, and it would have given me the actual value. Because it's exactly the same function to the right of 0 and to the left of 0. It's a reflection across the y-axis, so you get the same value there.

So you actually would've gotten 3, for this, multiplied by 2 and you get the value. If this one had diverged, that one also would have had to diverge, and the whole thing would have diverged. And that's because of the symmetry of the function over 0.

So I'm going to just briefly review what we did. And then we'll be done. So we come back over here. I gave you three integrals. We wanted to see if they converged or diverged, right, and if they converged find what they were. So the first one was cosine x. And we found that diverged for a different reason than what we've seen before. Because as x goes to infinity, the problem is the areas are varying up and down and they're bounded always. But the area under the curve is constantly changing and it's not converging to a fixed value. It's varying between minus 1 and 1 over and over again.

And then (b), we had this integral from 0 to 1 natural log x over root x dx. And the point here was a little more complicated, so let me come to that one. We used an integration by parts. We had an integral they converged easily, because this was an easy improper integral to determine. And then we had this other thing that we now have to evaluate it. You wound up having to use L'Hopital's rule to evaluate it.

But then you're able to show that this using L'Hopital's rule, this actually has a value at each endpoint, a finite value at each endpoint. It converges then to 0 as you, when you put in these bounds, and then you have a fixed value for this one when you take that integral. So you ended up with another integral that was improper. But you could show it converged. And then you could use L'Hopital's rule to find what happened at these endpoints. And you get a value of minus 4.

Then the third one was this integral minus 1 to 1 x to the minus 2/3 x dx. And the point I wanted to make there is that just because the function is well-behaved near the endpoints doesn't mean that it's still, you know, that everything is hunky-dory. You have to be careful and you have to check at places where the value of the function is going off to infinity.

So in the second case, the left endpoint posed a problem, and everything else was fine. In this case, the left and right endpoints are both fine, but in the middle there's a problem. And so you split it up into it's two pieces so that you have the endpoints. Now these two new integrals represent where the problem might happen, and then you do your, find your antiderivative, and evaluate, and see if you can get something that converges. So that's the idea of these types of problems. So hopefully that was helpful, and I think that's where I'll stop.