

Hi. Welcome back to recitation. We've been talking about Taylor series for a number of functions and rules by which you can compute Taylor series. I have here an example that I don't think we did in lecture. So this is the function f of x equals secant of x . Now, unlike some of the other ones you've seen, there's not a really simple formula for the whole Taylor series of secant x .

So what I'd like you to do is not to find, you know, a formula for the general term, but rather, just to use some of the tools that we've learned to compute the first few terms of the Taylor series for f of x equals secant x . Say, up through the x to the fourth term, if you wanted, or even a little further if you were feeling ambitious.

So why don't you pause the video, have a go at that, come back, and we can do it together.

So welcome back. I asked you to compute the first few terms of a Taylor series. One thing you can always do in this case, is you can go and you can apply the general formula that we have for Taylor series, and use it to compute the series that way. So in order to do that, you just need to compute a few derivatives of your function.

So remember that the general formula for a Taylor series is the Taylor series for f of x is equal to the sum from n equals 0 to infinity of the n th derivative of f taken at zero divided by n factorial times x to the n . And since we're only interested in the first few terms, this is f of 0 plus f prime of 0 x plus f double prime of 0 over $2 x$ squared, plus dot dot dot. I'm not going to write out the next few terms.

And if you want to apply this formula to secant of x , what you would have to do, is you would have to compute these derivatives. And so we could try doing that. So in our case, f of x is equal to secant of x . So f of 0 is secant of 0, which is just 1. So then we need to know f prime of x , so that's the derivative of secant of x . So this is one you should remember. This is equal to secant x times tan x . And when you plug in x equal to 0, well, the tan x is 0. So f prime of 0 is equal to 0.

And then you could keep going. So you could compute f double prime of x . So we would have to compute the derivative of secant x tan x . And you would do that by the product rule. You know, you take derivative of secant of x , and that gives you secant x tan x times tan x plus-- and then you, OK. So you leave secant of x , and you multiply it by the derivative of tangent, which is secant squared x . And now when you put in 0 here, you get f double prime of 0. Well, OK. So this has a tan x in it, so that part's going to give you 0. And here we end up with secant of 0 times secant squared of 0, so that's 1 times 1 times 1. So that's just 1. So f double prime of 0 is 1, and you could keep doing this.

Now one thing you'll notice is that this is getting more and more complicated. I mean, we can simplify this expression a little bit. We could write it as secant x tan squared x plus secant cubed x , and there are, you know, all

sorts of trig manipulations you could do, if you wanted to try and rewrite that in some simpler form. But fundamentally, it's more complicated than the first derivative was, and that's more complicated in the function. And it will keep getting more complicated as you compute more derivatives. So we can do this.

So so far this shows us, by the way, that this is equal to $1 + 0x + \frac{x^2}{2} + \dots$. And if you wanted to compute, you know, up through the fourth degree term or something like this, that's something that's manageable. But I want to suggest that there are maybe some nicer ways to do it.

So one thing to notice is that secant of x is closely related to the function cosine of x . And you know the Taylor series for cosine of x . So one thing you could think to do, is to leverage the information that you have about cosine of x in order to use it to get some information about secant of x . So one simple way to do that, is that you know that cosine of x is even. It's an even function. Cosine of minus x is equal to cosine of x . So that means secant of x is also even.

Secant of x is an even function. And you've seen that even functions, their Taylor series, all the odd powers have coefficient 0. So what that means is without ever computing the third derivative, we can know already that the next term in this Taylor series is going to be $0 \cdot \frac{x^3}{6}$. OK? So that's nice. So the odd terms of the Taylor series for secant of x are 0.

OK. So that's one thing you can get right away. So that gives you, if you like, that gives you half of the terms of the Taylor series. It's a little bit of a joke, but.

OK, so then you only need to figure out the even terms. That's one way we can leverage the relationship between secant and cosine. The other way is that we can remember that Taylor series multiply just like polynomials do. So if secant is $1/\cosine$, that means secant times cosine is equal to 1. OK? So idea, secant of x times cosine of x is equal to 1.

Now, that means that the Taylor series for secant of x times the Taylor series for cosine of x has to simplify just to 1. So we can write down that product as a product of two infinite polynomials, and we can start multiplying term by term. And that'll allow us to solve for a bunch of terms, just by solving some simple linear equations. So let me show you what I mean. So we know-- let me write it as cosine times secant. So we know that cosine of x is $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$

And so we know that if we multiply this by the series for secant of x -- well, what is the series for secant of x ? Well, we've already computed a few terms. We know that it's $1 + \frac{x^2}{2}$, we computed that already. And we know that the third degree term is 0. So there's some fourth degree coefficient, a_4 , x to the fourth, or 4

factorial. And there's some-- well, we know, we said it's even, so we know the fifth degree coefficient is 0. So then the sixth degree coefficient we don't know yet. So this is plus a sub 6 times x to the sixth over 6 factorial, plus dot dot dot.

So we know that when we multiply these two things together, it has to give us just 1. All the higher-order terms have to cancel, because over here we have a 1.

So what you can do, is you can actually try multiplying these things out. So it's easy to see, for example, that the constant term of this product is just 1 times 1, which is 1. Which is good. So that's a check on what we've done so far. And there is no x-term, because there are no x's. The x squared term here is 1 times x squared over 2 minus x squared over 2 times 1. Well, that gives us 0, so that's good. So this product is equal to-- well, it's 1, plus we saw 0x, plus 0 x squared. How about the x cubed term? Well, there are no odd terms in this product, so there's no x cubed. How about the x to the fourth term? Well, OK. So how do we get an x to the fourth? We could have an x to the fourth here times a constant. So that's x to the fourth over 24. Or, we could have an x squared times an x squared. So in this case, that gives us minus x squared over 2 times x squared over 2, which is minus x to the fourth over 4. Or we could have a constant times an x to the fourth. So this is plus a₄ x to the fourth over 24. And then we'll have a sixth-degree term, and so on.

Notice that there's never any involvement from the higher-order terms in the fourth power here. Right? If you ever took an x to the sixth here, well then everything you multiply it by has at least an x to the sixth power. So we don't have to worry about that showing up in the x to the fourth term.

Well, OK. We know this is actually equal to 1, so we know that this thing has to be 0. This is x to the fourth term. It has to be 0. So that means that $\frac{1}{24} - \frac{1}{4} + \frac{a_4}{24} = 0$. So OK. So now you can multiply everything through by 24 and rearrange to figure out that a₄ is equal to 5. So I've done that correctly.

And then if you wanted, it would be fairly easy to go back up and then you look at the x to the sixth term. And from there, you could figure out that a₆ was equal to, say, 61 or something like that. I think 61. And you could keep doing this.

So this is one way to compute more terms of the series for secant of x. Another thing you could do-- which I'll just mention very briefly, I'm not going to show you how to do it-- is that you can do long division on power series. So it actually works out very-- it works out just like this. It's mathematically equivalent. The way you actually do it looks different. It looks like long division. When you do long division with polynomials, you start with the highest-order term. Of course, power series don't have highest-order terms. What you actually do with a power series, is you start with the lowest-order term. So to divide this into 1, you'd say, oh, you need a factor of 1, and then you'll have a-- you know, you subtract off 1 times this, and that gives you an x squared plus 2. And so, OK, so you say, I need

a plus an x squared plus 2, and so on.

So that was too brief for you to understand it properly, but you can go and look up somewhere the method of long division on power series.

So just to recap, we talked about three methods for computing the coefficients of a power series. There's the formula that you were given, which works, and which you could use. In this case, it's a little complicated. Then there's the method of using a relationship between your power series and other known power series. In this case, we can use the relationship that we know our power series satisfies a certain product. We know that our power series times cosine of x equals to 1, and we can use that to solve for some of the coefficients. Or you can also, similarly, when you have that situation, you can also use long division to compute the coefficients.

I'll end there.