 Let me just tell you first about the list of topics. Basically, the list of topics is simple. It is everything. I mean, everything we have seen so far is on the exam. But let me just remind you of the main topics that we have seen. First of all, we learned about vectors, how to use them, and dot-product. At this point, you probably should know that the dot-product of two vectors is obtained by summing products of components. And geometrically it is the length of A times the length of B times the cosine of the angle between them. And, in particular, we can use dot-product to measure angles by solving for cosine theta in this equality. And most importantly to detect whether two vectors are perpendicular to each other. Two vectors are perpendicular when their dot-product is zero. Any questions about that? No. Is everyone reasonably happy with dot-product by now? I see a stunned silence. Nobody happy with dot-product so far? OK. If you want to look at Practice 1A, a good example of a typical problem with dot-product would be problem 1. Let's see. We are going to go over the practice exam when I am done writing this list of topics. I think probably we actually will skip this problem because I think most of you know how to do it. And if not then you should run for help from me or from your recitation instructor to figure out how to do it. The second topic that we saw was cross-product. When you have two vectors in space, you can just form that cross-product by computing its determinant. So, implicitly, we should also know about determinants. By that I mean two by two and three by three. Don't worry about larger ones, even if you are interested, they won't be on the test. And applications of cross-product, for example, finding the area of a triangle or a parallelogram in space. If you have a triangle in space with sides A and B then its area is one-half of the length of A cross B. Because the length of A cross B is length A, length B sine theta, which is the same as the area of the parallelogram formed by these two vectors. And the other application of cross-product is to find a vector that's perpendicular to two given vectors. In particular, to find the vector that is normal to a plane and then find the equation of a plane. Another application is finding the normal vector to a plane and using that finding the equation of a plane. Basically, remember, to find the equation of a plane, ax by cz = d, what you need is the normal vector to the plane. And the components of the normal vector are exactly the coefficients that go into this. And we have seen an argument for why that happens to be the case. To find a normal vector to a plane typically what we will do is take two vectors that lie in the plane and will take their cross-product. And the cross-product will automatically be perpendicular to both of them. We are going to see an example of that when we look at problem 5 in practice 1A. I think we will try to do that one. Another application, well, we will just mention it as a topic that goes along with this one. We have seen also about equations of lines and how to find where a line intersects a plane. Just to refresh your memories, the equation of a line, well, we will be looking at parametric equations. To know the parametric equation of a line, we need to know a point on the line and we need to know a vector that is parallel to the line. And, if we know a point on the line and a vector along the line, then we can express the parametric equations for the motion of a point that is moving on the line. Actually, starting at point, at time zero, and moving with velocity v. To put things in symbolic form, you will get a position of that point by starting with a position of time zero and adding t times the vector v. It gives you x, y and z in terms of t. And that is how we represent lines. We will look at problem 5 in a bit, but any general questions about these topics? No. Do you have a question? Do we have to know Taylor series? That is a good question. No, not on the exam. [APPLAUSE] Taylor series are something you should be aware of, generally speaking. It will be useful for you in real life, probably not when you go to the supermarket, but if you solve engineering problems you will need Taylor series. It would be good not to forget them entirely, but on the 18.02 exams they probably won't be there. Let me continue with more topics. And then we can see if you can think of other topics that should or should not be on the exam. Third topics would be matrices, linear systems, inverting matrices. I know that most of you that have calculators that can invert matrices, but still you are expected at this point to know how to do it by hand. If you have looked at the practice tests, both of them have a problem that asks you to invert a matrix or at least do part of it. And so it is very likely that tomorrow there will be a problem like that as well. In general, when a kind of problem is on both practice tests it's a good indication that it might be there also on the actual exam. Unfortunately, not with the same matrix so you cannot learn the answer by heart. Another thing that we have learned about, well, I should say this is going to be problem 3 on the test and will on the practice test. On the actual test, too, I think, actually. Anyway, we will come back to it later. A couple of things that you should remember. If you have a system of the form AX equals B then there are two cases. If a determinant of A is not zero then that means you can compute the inverse matrix and you can just solve by taking A inverse times B. And the other case is when the determinant of A is zero, and there is either no solution or there is infinitely many solutions. In particular, if you know that there is a solution, for example, if B is zero there always is an obvious solution, X equals zero, then you will actually have infinitely many. In general, we don't really know how to tell whether it is no solution or infinitely many. Questions about that? Yes? Will we have to know how to rotate vectors and so on? Not in general, but you might still want to remember how to rotate a vector in a plane by 90 degrees because that has been useful when we have done problems about parametric equations, which is what I am coming to next. What we have seen about rotation matrices, that was the homework part B problem, you are not supposed to remember by heart everything that was in part B of your homework. It is a good idea to have some vague knowledge because it is useful culture, I would say, useful background for later in your lives, but I won't ask you by heart to know what is the formation for a rotation matrix. And then we come to, last by not least, the problem of finding parametric equations. And, in particular, possibly by decomposing the position vector into a sum of simpler vectors. You have seen quite an evil exam of that on the last problem set with this picture that maybe by now you have had some nightmares about. Anyway, the one on the exam will certainly be easier than that. But, as you have seen -- I mean, you should know, basically, how to analyze a motion that is being described to you and express it in terms of vectors and then figure out what the parametric equation will be. Now, again, it won't be as complicated on the exam as the one in the problem set. But there are a couple of those on the practice exam, so that gives you an idea of what is realistically expected of you. And now once we have parametric equations for motion, so that means when we know how to find the position vector as a function of a parameter maybe of time, then we have seen also about velocity and acceleration, which the vector is obtained by taking the first and second derivatives of a position vector. And so one topic that I will add in there as well is somehow how to prove things about motions by differentiating vector identities. One example of that, for example, is when we try to look at Kepler's law in class last time. We look at Kepler's second law of planetary motion, and we reduced it to a calculation about a derivative of the cross-product R cross v. Now, on the exam you don't need to know the details of Kepler's law, but you need to be able to manipulate vector quantities a bit in the way that we did. And so on practice exam 1A, you actually have a variety of problems on this topics because you have problems two, four and six, all about parametric motions. Probably tomorrow there will not be three distinct problems about parametric motions, but maybe a couple of them. I think that is basically the list of topics. Anybody spot something that I have forgotten to put on the exam or questions about something that should or should not be there? You go first. Yeah? How about parametrizing weird trigonometric functions? I am not sure what you mean by that. Well, parametric curves, you need to know how to parameterize motions, and that involves a little bit of trigonometrics. When we have seen these problems about rotating wheels, say the cycloid, for example, and so on there is a bit of cosine and sine and so on. I think not much more on that. You won't need obscure trigonometric identities. You're next. Any proofs on the exam or just like problems? Well, a problem can ask you to show things. It is not going to be a complicated proof. The proofs are going to be fairly easy. If you look at practice 1A, the last problem does have a little bit of proof. 6B says that show that blah, blah, blah. But, as you will see, it is not a difficult kind of proof. So, about the same. Yes? Are there equations of 3D shapes that we should know at this point? We should know definitely a lot about the equations of planes on lines. And you should probably know that a sphere centered at the origin is the set of points where distance to the center is equal to the radius of the sphere. We don't need more at this point. As the semester goes on, we will start seeing cones and things like that. But at this point planes, lines. And maybe you need to know about circles and spheres, but nothing beyond that. More questions? Yes? If there is a formula that you have proved on the homework then, yes, you can assume it on the test. Maybe you want to write on your test that this is a formula you have seen in homework just so that we know that you remember it from homework and not from looking over your neighbor's shoulder or whatever. Yes, it is OK to use things that you know general-speaking. That being said, for example, probably there will be a linear system to solve. It will say on the exam you are supposed to solve that using matrices, not by elimination. There are things like that. If a problem says solve by using vector methods, things like that, then try to use at least a vector somewhere. But, in general, you are allowed to use things that you know. Yes? Will we need to go from parametric equations to xy equations? Well, let's say only if it is very easy. If I give you a parametric curve, sin t, sin t, then you should be able to observe that it is on the line y equals x, not beyond that. Yes? Do we have to use -- Yes. I don't know if you will have to use it, but certainly you should know a little bit about the unit tangent vector. Just remember the main thing to know that the unit tangent vector is velocity divided by the speed. I mean there is not much more to it when you think about it. Yes? Kepler's law, well, you are allowed to use it if it helps you, if you find a way to squeeze it in. You don't have to know Kepler's law in detail. You just have to know how to reproduce the general steps. If I tell you R cross v is constant, you might be expected to know what to do with that. I would say -- Basically, you don't need to know Kepler's law. You need to know the kind of stuff that we saw when we derived it such as how to take the derivative of a dot-product or a cross-product. That is basically the answer. I don't see any questions anymore. Oh, you are raising your hand. Yes. How to calculate the distance between two lines and the distance between two planes? Well, you have seen, probably recently, that it is quite painful to do in general. And, no, I don't think that will be on the exam by itself. You need to know how to compute the distance between two points. That certainly you need to know. And also maybe how to find the compliment of a vector in a certain direction. And that is about it, I would say. I mean the more you know about things the better. Things that come up on part Bs of the problem sets are interesting things, but they are usually not needed on the exams. If you have more questions then you are not raising your hand high enough for me to see it. OK. Let's try to do a bit of this practice exam 1A. Hopefully, everybody has it. If you don't have it, hopefully your neighbor has it. If you don't have it and your neighbor doesn't have it then please raise your hand. I have a couple. If you neighbor has it then just follow with them for now. I think there are a few people behind you over there. I will stop handing them out now. If you really need one, it is on the website, it will be here at the end of class. Let's see. Well, I think we are going to just skip problems 1 and 2 because they are pretty straightforward and I hope that you know how to do them. I mean I don't know. Let's see. How many of you have no problem with problem 1? How many of you have trouble with problem 1? OK. How many of you haven't raised your hands? OK. How many of you have trouble with problem 2? OK. Well, if you have questions about those, maybe you should just come see me at the end because that is probably more efficient that way. I am going to start right away with problem 3, actually. Problem 3 says we have a matrix given to us |1 3 2; 2 0 - 1; 1 1 0|. And it tells us determinant of A is 2 and inverse equals something, but we are missing two values A and B and we are supposed to find them. That means we need to do the steps of the algorithm to find the inverse of A. We are told that A inverse is one-half of |1 ... ...; - 1 - 2 5; 2 2 - 6|. And here there are two unknown values. Remember, to invert a matrix, first we compute the minors. Then we flip some signs to get the cofactors. Then we transpose. And, finally, we divide by the determinant. Let's try to be smart about this. Do we need to compute all nine minors? No. We only need to compute two of them, right? Which minors do we need to compute? Here and here or here and here? Yeah, that looks better. Because, remember, we need to transpose things so these two guys will end up here. I claim we should compute these two minors. And we will see if that is good enough. If you start doing others and you find that they don't end up in the right place then just do more, but you don't need to spend your time computing all nine of them. If you are worried about not doing it right then, of course, you can maybe compute one or two more to just double-check your answers. But let us just do those that we think are needed. The matrix of minors. The one that goes in the middle position is obtained by deleting this row and that column, and we are left with a determinant |3 2;1 0|, 3 times 0 minus 1 times 2 should be - 2 should be - 2. Then the one in the lower left corner, we delete the last row and the first column, we are left with |3 2; 0 - 1|. 3 times (- 1) is negative 3 minus 0. We are still left with negative three. Is that step clear for everyone? Then we need to go to cofactors. That means we need to change signs. The rule is -- We change signs in basically these four places. That means we will be left with positive 2 and negative 3. Then we take the transpose. That means the first column will copy into the first row, so this guy we still don't know, but here we will have two and here we will have minus three. Finally, we have to divide by the determinant of A. And here we are actually told that the determinant of A is two. So we will divide by two. But there is only one-half here so actually it is done for us. The values that we will put up there are going to be 2 and negative 3. Now let's see how we use that to solve a linear system. If we have to solve a linear system, Ax equals B, well, if the matrix is invertible, its determinant is not zero, so we can certainly write x equals A inverse B. So we have to multiply, that is one-half | 1 2 - 3; - 1 - 2 5; 2 2 - 6|. Times B [1, - 2,1]. Remember, to do a matrix multiplication you take the rows in here, the columns in here and you do dot-products. The first entry will be one times one plus two times minus two plus minus three times one, one minus four minus three should be negative six, except I still have, of course, a one-half in front. Then minus one plus four plus five should be 8. Two minus four minus six should be -8. That will simplify to [- 3,4, - 5]. Any questions about that? OK. Now we come to part C which is the harder part of this problem. It says let's take this matrix A and let's replace the two in the upper right corner by some other number C. That means we will look at 1 3 C; 2 0 - 1; 1 1 0|. And let's call that M. And it first asks you to find the value of C for which this matrix is not invertible. M is not invertible exactly when the determinant of M is zero. Let's compute the determinant. Well, we should do one times that smaller determinant, which is zero minus negative one, which is 1 times 1 minus three times that determinant, which is zero plus one is 1. And then we have plus C times the lower left determinant which is two times one minus zero is 2. That gives us one minus three is - 2 2C. That is zero when C equals 1. For C equals 1, this matrix is not invertible. For other values it is invertible. It goes on to say let's look at this value of C and let's look at the system Mx equals zero. I am going to put value one in there. Now, if we look at Mx equals zero, well, this has either no solution or infinitely many solutions. But here there is an obvious solution. Namely x equals zero is a solution. Maybe let me rewrite it more geometrically. X 3 y z = 0. 2x - z = 0. And x y = 0. You see we have an obvious solution, (0,0, 0). But we have more solutions. How do we find more solutions? Well, (x, y, z) is a solution if it is in all three of these planes. That is a way to think about it. Probably we are actually in this situation where, in fact, we have three planes that are all passing through the origin and all parallel to the same line. And so that would be the line of solutions. To find it actually we can think of this as follows. The first observation is that actually in this situation we don't need all three equations. The fact that the system has infinitely many solutions means that actually one of the equations is redundant. If you look at it long enough you will see, for example, if you multiply three times this equation and you subtract that one then you will get the first equation. Three times (x y) - (2x - z) will be x 3y z. Now, we don't actually need to see that to solve a problem. I am just showing you that is what happens when you have a matrix with determinant zero. One of the equations is somehow a duplicate of the others. We don't actually need to figure out how exactly. What that means is really we want to solve, let's say start with two of the equations. To find the solution we can observe that the first equation says actually that dot-product with =0. And the second equation says dot-product with <2,0,- 1> is zero. And the third equation, if we really want to keep it, says we should be also having this. Now, these equations now written like this, they are just saying we want an x, y, z that is perpendicular to these vectors. Let's forget this one and let's just look at these two. They are saying we want a vector that is perpendicular to these two given vectors. How do we find that? We do the cross-product. To find x, y, z perpendicular to <1,3, 1> and <2,0, - 1>, we take the cross-product. And that will give us something. Well, let me just give you the answer. I am sure you know how to do cross-products by now. I don't have the answer here, so I guess I have to do it. That should be <- 3, probably positive 3, and then - 6>. That is the solution. And any multiple of that is a solution. If you like to neatly simplify them you could say negative one, one, negative two. If you like larger numbers you can multiply that by a million. That is also a solution. Any questions about that? Yes? That is correct. If you pick these two guys instead, you will get the same solution. Well, up to a multiple. It could be if you do the cross-product of these two guys you actually get something that is a multiple -- Actually, I think if you do the cross-product of the first and third one you will get actually minus one, one, minus two, the smaller one. But it doesn't matter. I mean it is really in the same direction. This is all because a plane has actually normal vectors of all sizes. Yes? I don't think so because -- An important thing to remember about cross-product is we compute for minors, but then we put a minus sign on the second component. The coefficient of j in here, the second component, you do one times minus two times one. That is negative three indeed. But then you actually change that to a positive three. Yes? Well, we don't have parametric equations here. Oh, solving by elimination. Well, if it says that you have to use vector methods then you should use vector methods. If it says you should use vectors and matrices then you are expected to do it that way. Yes? It depends what the problem is asking. The question is, is it enough to find the components of a vector or do we have to find the equation of a line? Here it says find one solution using vector operations. We have found one solution. If you wanted to find the line then it would all the things that are proportional to this. It would be maybe minus 3t, 3t minus 6t, all the multiples of that vector. We do because (0,0, 0) is an obvious solution. Maybe I should write that on the board. You had another question? Not quite. Let me re-explain first how we get all the solutions and why I did that cross-product. First of all, why did I take that cross-product again? I took that cross-product because I looked at my three equations and I observed that my three equations can be reformulated in terms of these dot-products saying that x, y, z is actually perpendicular these guys and these guys have normal vectors to the planes. Remember, to be in all three planes it has to be perpendicular to the normal vectors. That is how we got here. And now, if we want something that is perpendicular to a bunch of given vectors, well, to be perpendicular to two vectors, an easy way to find one is to take that cross-product. And, if you take any two of them, you will get something that is the same up to scaling. Now, what it means geometrically is that when we have our three planes and they all actually contain the same line -- And we know that is actually the smae case because they all pass through the origin. They pass through the origin because the constant terms are just zero. What happens is that the normal vectors to these planes are, in fact, all perpendicular to that line. The normal vectors -- Say this line is vertical. The normal vectors are all going to be horizontal. Well, it is kind of hard to draw. By taking the cross-product between two normal vectors we found this direction. Now, to find actually all the solutions. What we know so far is that we have this direction <-3 3 - 6>. That is going to be parallel to the line of intersections. Let me do it here, for example, . Now we have one particular solution. 0,0, 0. Actually, we have found another one, too, which is <- 3,3, - 6>. Anyway, if a line of solutions -- -- has parametric equation x = - 3t, y = 3t, z = - 6t, anything proportional to that. That is how we would find all the solutions if we wanted them. It is almost time. I think I need to jump ahead to other problems. Let's see. I think problem 4 you can probably find for yourselves. It is a reasonably straightforward parametric equation problem. You just have to find the coordinates of point P. And for that it is a very simple trick. Problem 5. Find the area of a spaced triangle. It sounds like a cross-product. Find the equation of a plane also sounds like a cross-product. And find the intersection of this plane with a line means we find first the parametric equation of the line and then we plug that into the equation of the plane to get where they intersect. Does that sound reasonable? Who is disparate about problem 5? OK. Let me repeat problem 5. First part we need to find the area of a triangle. And the way to do that is to just do one-half the length of a cross-product. If we have three points, P0, P1, P2 then maybe we can form vectors P0P1 and P0P2. And, if we take that cross-product and take the length of that and divide by two, that will give us the area of a triangle. Here it turns out that this guy is , if I look at the solutions, so you will end up with square root of 6 over 2. The second is asking you for the equation of a plane containing these three points. Well, first of all, we know that a normal vector to the plane is going to be given by this cross-product again. That means that the equation of plane will be of a form x plus y plus 2z equals something. If a coefficient is here it comes from the normal vector. And to find what goes in the right-hand side, we just plug in any of the points. If you plug in P0, which is (2,1, 0) then two plus one seems like it is 3. And, if you want to double-check your answer, you can take P1 and P2 and check that you also get three. It is a good way to check your answer. Then the third part. We have a line parallel to the vector v equals one, one, one through the point S, which is (- 1,0, 0). That means you can find its parametric equation. X will start at - 1, increases at rate 1. Y starts at zero, increases at rate one. Z starts at zero, increases at rate one. You plug these into the plane equation, and that will tell you where they intersect. Is that clear? And now, in the last one minute, on that side I have one minute, let me just say very quickly -- Well, do you want to hear about problem 6 anyway very quickly? Yeah. OK. Problem 6 is one of these like vector calculations. It says we have a position vector R. And it asks you how do we find the derivative of R dot R? Well, remember we have a product rule for taking the derivative. UV prime is U prime V plus UV prime. It also applies for dot-product. That is dR by dt dot R plus R dot dR by dt. And these are both the same thing. You get two R dot dR/dt, but dR/dt is v for velocity vector. Hopefully you have seen things like that. Now, it says show that if R has constant length then they are perpendicular. All you need to write basically is we assume length R is constant. That is what it says, R has constant length. Well, how do we get to, say, something we probably want to reduce to that? Well, if R is constant in length then R dot R is also constant. And so that means d by dt of R dot R is zero. That is what it means to be constant. And so that means R dot v is zero. That means R is perpendicular to v. That is a proof. It is not a scary proof. And then the last question of the exam says let's continue to assume that R has constant length, and let's try to find R dot v. If there is acceleration then probably we should bring it in somewhere, maybe by taking a derivative of something. If we know that R dot v equals zero, let's take the derivative of that. That is still zero. But now, using the product rule, dR/dt is v dot v plus R dot dv/dt is going to be zero. That means that you are asked about R dot A. Well, that is equal to minus V dot V. And that is it.