

## 4. Line Integrals in the Plane

### 4A. Plane Vector Fields

**4A-1** Describe geometrically how the vector fields determined by each of the following vector functions looks. Tell for each what the largest region in which  $\mathbf{F}$  is continuously differentiable is.

a)  $a\mathbf{i} + b\mathbf{j}$ ,  $a, b$  constants      b)  $-x\mathbf{i} - y\mathbf{j}$       c)  $\frac{x\mathbf{i} + y\mathbf{j}}{r}$       d)  $\frac{y\mathbf{i} - x\mathbf{j}}{r}$

**4A-2** Write down the gradient field  $\nabla w$  for each of the following:

a)  $w = ax + by$       b)  $w = \ln r$       c)  $w = f(r)$

**4A-3** Write down an explicit expression for each of the following fields:

- Each vector has the same direction and magnitude as  $\mathbf{i} + 2\mathbf{j}$ .
- The vector at  $(x, y)$  is directed radially in towards the origin, with magnitude  $r^2$ .
- The vector at  $(x, y)$  is tangent to the circle through  $(x, y)$  with center at the origin, clockwise direction, magnitude  $1/r^2$ .
- Each vector is parallel to  $\mathbf{i} + \mathbf{j}$ , but the magnitude varies.

**4A-4** The electromagnetic force field of a long straight wire along the  $z$ -axis, carrying a uniform current, is a two-dimensional field, tangent to horizontal circles centered along the  $z$ -axis, in the direction given by the right-hand rule (thumb pointed in positive  $z$ -direction), and with magnitude  $k/r$ . Write an expression for this field.

### 4B. Line Integrals in the Plane

**4B-1** For each of the fields  $\mathbf{F}$  and corresponding curve  $C$  or curves  $C_i$ , evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ .

Use any convenient parametrization of  $C$ , unless one is specified. Begin by writing the integral in the differential form  $\int_C M dx + N dy$ .

- $\mathbf{F} = (x^2 - y)\mathbf{i} + 2x\mathbf{j}$ ;  $C_1$  and  $C_2$  both run from  $(-1, 0)$  to  $(1, 0)$ :  
 $C_1$ : the  $x$ -axis       $C_2$ : the parabola  $y = 1 - x^2$
- $\mathbf{F} = xy\mathbf{i} - x^2\mathbf{j}$ ;  $C$ : the quarter of the unit circle running from  $(0, 1)$  to  $(1, 0)$ .
- $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ ;  $C$ : the triangle with vertices at  $(0, 0)$ ,  $(0, 1)$ ,  $(1, 0)$ , oriented clockwise.
- $\mathbf{F} = y\mathbf{i}$ ;  $C$  is the ellipse  $x = 2\cos t$ ,  $y = \sin t$ , oriented counterclockwise.
- $\mathbf{F} = 6y\mathbf{i} + x\mathbf{j}$ ;  $C$  is the curve  $x = t^2$ ,  $y = t^3$ , running from  $(1, 1)$  to  $(4, 8)$ .
- $\mathbf{F} = (x + y)\mathbf{i} + xy\mathbf{j}$ ;  $C$  is the broken line running from  $(0, 0)$  to  $(0, 2)$  to  $(1, 2)$ .

**4B-2** For the following fields  $\mathbf{F}$  and curves  $C$ , evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  without any formal calculation, appealing instead to the geometry of  $\mathbf{F}$  and  $C$ .

- $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ ;  $C$  is the counterclockwise circle, center at  $(0, 0)$ , radius  $a$ .
- $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ ;  $C$  is the counterclockwise circle, center at  $(0, 0)$ , radius  $a$ .

**4B-3** Let  $\mathbf{F} = \mathbf{i} + \mathbf{j}$ . How would you place a directed line segment  $C$  of length one so that the value of  $\int_C \mathbf{F} \cdot d\mathbf{r}$  would be

- a) a maximum;    b) a minimum;    c) zero;  
 d) what would the maximum and minimum values of the integral be?

#### 4C. Gradient Fields and Exact Differentials

**4C-1** Let  $f(x, y) = x^3y + y^3$ , and  $C$  be  $y^2 = x$ , between  $(1, -1)$  and  $(1, 1)$ , directed upwards.

- a) Calculate  $\mathbf{F} = \nabla f$ .  
 b) Calculate the integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  three different ways:  
 (i) directly;  
 (ii) by using path-independence to replace  $C$  by a simpler path  
 (iii) by using the Fundamental Theorem for line integrals.

**4C-2** Let  $f(x, y) = xe^{xy}$ , and  $C$  be the path  $y = 1/x$  from  $(1, 1)$  to  $(0, \infty)$ .

- a) Calculate  $\mathbf{F} = \nabla f$ .  
 b) Calculate the integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$   
 (i) directly;  
 (ii) by using the Fundamental Theorem for line integrals.

**4C-3** Let  $f(x, y) = \sin x \cos y$ .

- a) Calculate  $\mathbf{F} = \nabla f$ .  
 b) What is the maximum value  $\int_C \mathbf{F} \cdot d\mathbf{r}$  can have over all possible paths  $C$  in the plane? Give a path  $C$  for which this maximum value is attained.

**4C-4\*** The Fundamental Theorem for line integrals should really be called the First Fundamental Theorem. There is an analogue for line integrals of the Second Fundamental Theorem also, where you first integrate, then differentiate; it provides the justification for Method 1 in this section. It runs:

If  $\int_C M dx + N dy$  is path-independent, and  $f(x, y) = \int_{(x_0, y_0)}^{(x, y)} \mathbf{F} \cdot d\mathbf{r}$ , then  $\nabla f = M \mathbf{i} + N \mathbf{j}$ .

The conclusion says  $f_x = M, f_y = N$ ; prove the second of these. (Hint: use the Second Fundamental Theorem of Calculus.)

**4C-5** For each of the following, tell for what value of the constants the field will be a gradient field, and for this value, find the corresponding (mathematical) potential function.

- a)  $\mathbf{F} = (y^2 + 2x) \mathbf{i} + axy \mathbf{j}$                       b)  $\mathbf{F} = e^{x+y}((x+a) \mathbf{i} + x \mathbf{j})$

**4C-6** Decide which of the following differentials is exact. For each one that is exact, express it in the form  $df$ .

- a)  $y dx - x dy$                                       b)  $y(2x + y) dx + x(2y + x) dy$   
 c)\*  $x \sin y dx + y \sin x dy$                       d)\*  $\frac{y dx - x dy}{(x + y)^2}$

### 4D. Green's Theorem

**4D-1** For each of the following fields  $\mathbf{F}$  and closed positively oriented curves  $C$ , evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  both directly, as a line integral, and also by applying Green's theorem and calculating a double integral.

- a)  $\mathbf{F} : 2y\mathbf{i} + x\mathbf{j}$ ,  $C : x^2 + y^2 = 1$
- b)  $\mathbf{F} : x^2(\mathbf{i} + \mathbf{j})$   $C$ : rectangle joining  $(0,0)$ ,  $(2,0)$ ,  $(0,1)$ ,  $(2,1)$
- c)  $\mathbf{F} : xy\mathbf{i} + y^2\mathbf{j}$ ,  $C : y = x^2$  and  $y = x$ ,  $0 \leq x \leq 1$

**4D-2** Show that  $\oint_C 4x^3y dx + x^4 dy = 0$  for all closed curves  $C$ .

**4D-3** Find the area inside the hypocycloid  $x^{2/3} + y^{2/3} = 1$ , by using Green's theorem. (This curve can be parametrized by  $x = \cos^3 \theta$ ,  $y = \sin^3 \theta$ , between suitable limits on  $\theta$ .)

**4D-4** Show that the value of  $\oint_C -y^3 dx + x^3 dy$  around any positively oriented simple closed curve  $C$  is always positive.

**4D-5** Show that the value of  $\oint_C xy^2 dx + (x^2y + 2x) dy$  around any square  $C$  in the  $xy$ -plane depends only on the size of the square, and not upon its position.

**4D-6\*** Show that  $\oint_C -x^2y dx + xy^2 dy > 0$  around any simple closed curve  $C$ .

**4D-7\*** Show that the value of  $\oint_C y(y+3) dx + 2xy dy$  around any equilateral triangle  $C$  depends only on the size of the triangle, and not upon its position in the  $xy$ -plane.

### 4E. Two-dimensional Flux

**4E-1** Let  $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ . Recalling the interpretation of this field (example 4, V1.5), or just by remembering how it looks geometrically, evaluate with little or no calculation the flux integral  $\int_C \mathbf{F} \cdot \mathbf{n} ds$ , where

- a)  $C$  is a circle of radius  $a$  centered at  $(0,0)$ , directed counterclockwise.
- b)  $C$  is the line segment running from  $(-1,0)$  to  $(1,0)$
- c)  $C$  is the line running from  $(0,0)$  to  $(1,0)$ .

**4E-2** Let  $\mathbf{F}$  be the constant vector field  $\mathbf{i} + \mathbf{j}$ . Where would you place a directed line segment  $C$  of length one in the plane so that the flux across  $C$  would be

a) maximal   b) minimal   c) zero   d)  $-1$    e) what would the maximal and minimal values be?

**4E-3** Let  $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$ . Evaluate  $\int_C \mathbf{F} \cdot \mathbf{n} ds$  if  $C$  is given by  $r(t) = (t+1)\mathbf{i} + t^2\mathbf{j}$ , where  $0 \leq t \leq 1$ ; the positive direction on  $C$  is the direction of increasing  $t$ .

**4E-4** Take  $C$  to be the square of side 1 with opposite vertices at  $(0,0)$  and  $(1,1)$ , directed clockwise. Let  $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ ; find the flux across  $C$ .

**4E-5** Let  $\mathbf{F}$  be defined everywhere except at the origin by the description:

dir  $\mathbf{F}$  = radially outward,  $|\mathbf{F}| = r^m$ ,  $m$  an integer.

a) Evaluate the flux of  $\mathbf{F}$  across a circle of radius  $a$  and center at the origin, directed counterclockwise.

b) For which value(s) of  $m$  will the flux be independent of  $a$ ?

**4E-6\*** Let  $\mathbf{F}$  be a constant vector field, and let  $C$  be a closed polygon, directed counterclockwise. Show that  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ . (Hint evaluate the integral along one of the directed sides; then add up the integrals over the successive sides, using properties of vectors.)

**4E-7\*** Let  $\mathbf{F}$  be a constant vector field, and  $C$  a closed polygon, as in the preceding exercise. Show that  $\oint_C \mathbf{F} \cdot \mathbf{n} ds = 0$ .

#### 4F. Green's Theorem in Normal Form

**4F-1** Calculate the functions  $\text{div } \mathbf{F}$  and  $\text{curl } \mathbf{F}$  for each of the following fields.

a)  $a\mathbf{i} + b\mathbf{j}$  ( $a, b$  constants)      b)  $x^2\mathbf{i} + y^2\mathbf{j}$       c)  $xy(\mathbf{i} + \mathbf{j})$

**4F-2** Let  $\mathbf{F} = \omega(-y\mathbf{i} + x\mathbf{j})$  be the vector field of Section V1, Example 4.

a) Calculate  $\text{div } \mathbf{F}$  and  $\text{curl } \mathbf{F}$ .

b) Using the physical interpretation of this vector field, explain why it is reasonable that  $\text{div } \mathbf{F} = 0$ .

c) Using the physical interpretation of  $\text{curl } \mathbf{F}$ , explain why it is reasonable that  $\text{curl } \mathbf{F} = 2\omega$  at the origin.

**4F-3** Verify Green's theorem in the normal form by calculating both sides and showing they are equal if  $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ , and  $C$  is formed by the upper half of the unit circle and the  $x$ -axis interval  $[-1, 1]$ .

**4F-4** Verify Green's theorem in the normal form by calculating both sides and showing they are equal if  $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$ , and  $C$  is the square with opposite vertices at  $(0, 0)$  and  $(1, 1)$ .

**4F-5** Calculate  $\text{div } \mathbf{F}$  and  $\text{curl } \mathbf{F}$  for  $\mathbf{F} = r^n(x\mathbf{i} + y\mathbf{j})$ . (Simplify the differentiation by using  $r_x = x/r$ ,  $r_y = y/r$ .)

For which value(s) of  $n$  is  $\text{div } \mathbf{F} = 0$ ? For which value(s) of  $n$  is  $\text{curl } \mathbf{F} = 0$ ?

**4F-6\*** a) Suppose that all the vectors of a field  $\mathbf{F}$  point radially outward and their magnitude is a differentiable function  $f(r)$  of  $r$  alone. Show that  $\text{curl } \mathbf{F} = 0$ .

b) Suppose all the vectors of a field  $\mathbf{F}$  are parallel. Reasoning from the physical interpretation of  $\text{curl } \mathbf{F}$ , would you expect it to be zero everywhere? Illustrate your answer by an example.

**4G. Simply-connected Regions.**

**4G-1** Using the criterion of this section, tell which of the following fields and differentials definitely are respectively conservative or exact, which of them are definitely not, and for which of them the criterion fails.

- a)  $(y^2 + 2)\mathbf{i} + 2xy\mathbf{j}$       b)  $x(\cos y) dx + y(\cos x) dy$   
 c)  $\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{r^2 - 1}}$       d)  $\frac{x dx + y dy}{\sqrt{1 - r^2}}$       e)  $\sqrt{x}\mathbf{i} + \sqrt{y}\mathbf{j}$

**4G-2** For each of the following fields  $\mathbf{F}$ , find  $f(x, y)$  such that  $\mathbf{F} = \nabla f$ .

- a) the field of 1a      b) the field of 1e  
 c) the field of 1d; use polar coordinates.

**4G-3** Evaluate  $\int_{(1,1)}^{(3,4)} \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{r^3}$ . Use the results of Example 3 in Notes V5.

**4G-4** Even though the field of Example 1,  $\mathbf{F} = xy\mathbf{i} + x^2\mathbf{j}$ , is not a gradient field, show that  $\oint \mathbf{F} \cdot d\mathbf{r} = 0$  around every simple closed curve which is symmetric about the  $y$ -axis.

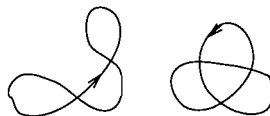
**4G-5** Which of the following regions are simply-connected?

- a) the half-plane lying above the  $x$ -axis  
 b) the plane minus the line segment joining  $(0,0)$  and  $(0,1)$   
 c) the plane minus the positive  $x$ -axis  
 d) the plane minus the entire  $x$ -axis  
 e) in polar coordinates, the region where  $r > 0$ ,  $0 < \theta < \theta_0$   
 f) the region between two concentric circles  
 g) the region in the plane between the two branches of the hyperbola  $xy = 1$

**4G-6** For which of the following vector fields is the domain where it is defined and continuously differentiable a simply-connected region?

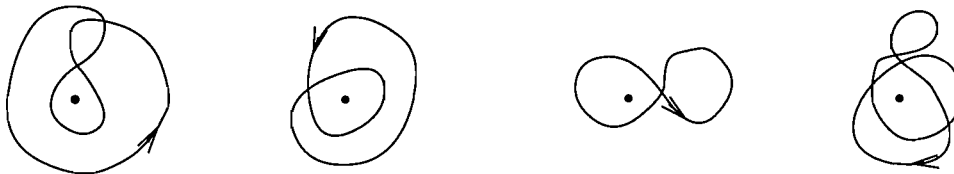
- a)  $\sqrt{x}\mathbf{i} + \sqrt{y}\mathbf{j}$       b)  $\frac{\mathbf{i} + \mathbf{j}}{\sqrt{1 - x^2 - y^2}}$       c)  $\frac{\mathbf{i} + \mathbf{j}}{\sqrt{x^2 + y^2 - 1}}$   
 d)  $\frac{-y\mathbf{i} + x\mathbf{j}}{r}$       e)  $(\mathbf{i} + \mathbf{j})\ln(x^2 + y^2)$

**4G-7\*** By following the method outlined in the proof of (3), show that if  $\text{curl } \mathbf{F} = 0$  in the whole  $xy$ -plane, then  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$  over each of the following closed paths (break them into as few pieces as possible):

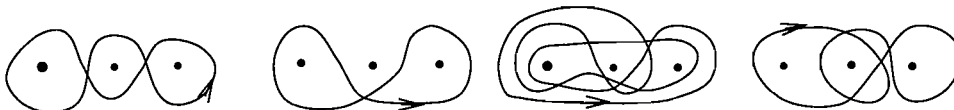


### 4H. Multiply-connected Regions

**4H-1** For each of the closed curves shown determine the winding number about the indicated point. Then assume the point is  $(0, 0)$ , and tell what the value of  $\oint_C \mathbf{F} \cdot d\mathbf{r}$  around  $C$  is, where  $\mathbf{F}$  is the vector field (1).



**4H-2** Suppose  $\mathbf{F}$  is continuous differentiable everywhere in the  $xy$ -plane except at the three points  $-1, 0, 1$  on the  $x$ -axis. Suppose the line integrals of  $\mathbf{F}$  around small circles surrounding each of these points have respectively the values  $2, \sqrt{2}$ , and  $\sqrt{3}$ . What is the value of  $\oint_C \mathbf{F} \cdot d\mathbf{r}$  around each of the following closed curves?



### 4I. Laplace's Equation and Harmonic Functions

**4I-1\*** Suppose  $f(x, y)$  is a polynomial in  $x$  and  $y$  all of whose terms have the same total degree  $n$  in  $x$  and  $y$  (such a polynomial is called "homogeneous of degree  $n$ "). Show that,  $a$  and  $b$  being arbitrary constants,

a) if  $n = 2$ ,  $f(x, y)$  is harmonic  $\Leftrightarrow f = a(x^2 - y^2) + bxy$

b) if  $n = 3$ ,  $f(x, y)$  is harmonic  $\Leftrightarrow f = a(x^3 - 3xy^2) + b(3x^2y - y^3)$ .

In each case, prove  $\Leftarrow$  by using (5).

**4I-2\*** a) Show that the functions  $c \ln r$  are harmonic ( $c$  constant; use  $r_x = x/r$ , etc.)

b) Show that if  $w = f(r)$  and  $w$  is harmonic, then  $f(r) = c_1 \ln r + c_2$ .

(Hint: introduce a new variable  $y = dw/dr$  and solve the resulting differential equation in  $y$  and  $r$ .)

**4I-3\*** Verify that  $e^{kx} \sin(ky + c)$  is harmonic for any constants  $k$  and  $c$ .

**4I-4\*** Can you find a function  $\phi$  harmonic in a region containing the circle of radius 1 and center at the origin, if the boundary values are prescribed by the following ( $s$  is the arclength along the circle, measured counterclockwise from  $(1, 0)$ ):

a)  $\frac{\partial \phi}{\partial \eta} = \cos s$     b)  $\frac{\partial \phi}{\partial \eta} = s^2 - 2\pi s$     c)  $\phi(s) = s$ .

**4I-5\*** "Derive" the equation beneath (12) symbolically, assuming that the product rule for differentiation applies to the operator  $\nabla$ .

**4I-6\*** The function  $\ln r$  is zero on the unit circle and harmonic; why doesn't this contradict Theorem 3, (13)?

**4I-7\*** Prove the **reciprocity law**: if  $\phi$  and  $\psi$  are harmonic in a region containing a simple closed curve  $C$  and its interior  $R$ , then

$$\oint_C \psi \frac{\partial \phi}{\partial \eta} ds = \oint_C \phi \frac{\partial \psi}{\partial \eta} ds.$$

(Hint: use Theorem 2.)

**4I-8\*** An important property of a harmonic function is that its value at the center of any circle is equal to its average value over the circle. (This is the basis of one important numerical method for calculating it by successive approximations.)

What would be the value of  $\phi(0, 0)$ , if on the unit circle  $\phi(s) = \sin^2 s$ ? (cf. Exercise 4 for the notations.)