MODEL ANSWERS TO HWK #8
(18.022 FALL 2010)

(1) (4.2.1) (a) $\nabla f(x,y) = (4-2x, 6-2y) = (0,0) \Rightarrow (x,y) = (2,3)$.
(b) $f(2+s, 3+t) - f(2,3) = -s^2 - t^2 < 0$ for all s, t. $\therefore$ $(2,3)$ is the maximum point.
(c) $Hf(2,3) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$. $d_1 = -2 < 0$ and $d_2 = 4 > 0$, hence it is negative definite. So $(2,3)$ is locally maximum.

(2) (4.2.6) $\nabla f(x,y) = (-2y^2 + 3x^2 - 1, 4y^3 - 4xy) = (0,0)$. Therefore $y^3 = xy$. If $y = 0$, then $x = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$. If $y \neq 0$, then $y^2 = x$. So $3x^2 - 2x - 1 = 0$ and $x = 1, -\frac{1}{3}$. But since $x = y^2 \geq 0, x = 1$. So the critical points are $(\frac{1}{\sqrt{3}}, 0)$, $(\frac{-1}{\sqrt{3}}, 0)$, $(1,1)$ and $(1,-1)$. Since the Hessian is $Hf(x,y) = \begin{pmatrix} 6x & -4y \\ -4y & 12y^2 - 4x \end{pmatrix}$,

- at $(\frac{1}{\sqrt{3}}, 0)$: $Hf = \begin{pmatrix} 2\sqrt{3} & 0 \\ 0 & \frac{-4}{\sqrt{3}} \end{pmatrix}$. Saddle point.
- at $(\frac{-1}{\sqrt{3}}, 0)$: $Hf = \begin{pmatrix} -2\sqrt{3} & 0 \\ 0 & \frac{4}{\sqrt{3}} \end{pmatrix}$. Saddle point.
- at $(1,1)$: $Hf = \begin{pmatrix} 6 & -4 \\ -4 & 8 \end{pmatrix}$. Local minimum.
- at $(1,-1)$: $Hf = \begin{pmatrix} 6 & 4 \\ 4 & 8 \end{pmatrix}$. Local minimum.

(3) (4.2.8) $\nabla f(x,y) = (e^x \sin y, e^x \cos y) = (0,0)$. Since $e^x \neq 0$ for all x, we have $\sin y = \cos y = 0$. But there's no such y. So there's no critical point.

(4) (4.2.22) (a) $\nabla f(x,y) = (2kx - 2y, -2x + 2ky) = (0,0)$ at $(0,0)$, so it's a critical point. $Hf(0,0) = \begin{pmatrix} 2k & -2 \\ -2 & 2k \end{pmatrix}$, and $d_1 = 2k$, $d_2 = 4k^2 - 4$. So (0,0) is a nondegenerate local minimum (i.e. the Hessian is positive definite) iff $k > 1$. It is local maximum (i.e. the Hessian is negative definite) iff $k < -1$.
(b) $\nabla g(x,y,z) = (2kx + kz, -2z - 2y, kx - 2y + kz) = (0,0,0)$ at $(0,0,0)$, so it's a critical point. $Hf(0,0,0) = \begin{pmatrix} 2k & 0 & k \\ 0 & -2 & -2 \\ k & -2 & k \end{pmatrix}$, and $d_1 = 2k$, $d_2 = -4k$, $d_3 = -2k^2 - 8k$. So (0,0,0) is a nondegenerate local maximum (i.e. the Hessian is negative definite) iff $k < -4$. On the other hand, (0,0,0) cannot be a nondegenrate local minimum (i.e. the Hessian is positive definite).

(5) (4.2.23) (a) $\nabla f(x,y) = (2ax, 2by) = (0,0) \Rightarrow (x,y) = (0,0)$. So the origin is the only critical point. $Hf(0,0) = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$ is positive definite iff $a > 0, b > 0$, and negative definite iff

$a < 0, b < 0$. So the origin is a local minimum if $a, b > 0$, local maximum if $a, b < 0$, and saddle point otherwise.

(b) $\nabla f(x, y, z) = (2ax, 2by, 2cz) = (0, 0, 0) \Rightarrow (x, y, z) = (0, 0, 0)$. So the origin is the only critical point. $Hf(0, 0, 0) = \begin{pmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{pmatrix}$ is positive definite iff $a > 0, b > 0, c > 0$, and negative definite iff $a < 0, b < 0, c < 0$. So the origin is a local minimum if $a, b, c > 0$, local maximum if $a, b, c < 0$, and saddle point otherwise.

(c) The very same argument as in (a) and (b) says the origin is the only critical point. Also the Hessian is the diagonal matrix with $2a_i$ at each i-th diagonal entry. Clearly it is positive definite iff all a_i are positive, and negative definite iff all a_i are negative. So the origin is a local minimum if all a_i are positive, local maximum if all a_i are negative, saddle point otherwise.

(6) (4.2.33) Solve $\nabla f(x, y) = (\cos x \cos y, -\sin x \sin y) = (0, 0)$ where $0 < x < 2\pi$ and $0 < y < 2\pi$. If $\cos x = 0$ then $\sin x \neq 0$, so $\sin y = 0$, and $(x, y) = (\pi/2, \pi), (3\pi/2, \pi)$. If $\cos x \neq 0$ then $\cos y = 0$, so $\sin y \neq 0$ and $\sin x = 0$. So $(x, y) = (\pi, \pi/2), (\pi, 3\pi/2)$. Evaluating f at each of these critical points, we get $f(\pi/2, \pi) = -1, f(3\pi/2, \pi) = 1, f(\pi, \pi/2) = f(\pi, 3\pi/2) = 0$. Now look at the boundaries. If $x = 0$ or $x = 2\pi$, then $f(x, y) = 0$. If $y = 0$ or $y = 2\pi$, then $f(x, y) = \sin x$, hence the maximum is 1 when $x = \pi/2$ and the minimum is -1 when $x = 3\pi/2$. Therefore comparing all the values, we conclude that the absolute maximum value of f is 1, and the absolute minimum value of f is -1 in R. (Actually in this problem, if one notices that f cannot be greater than 1 or less than -1, just finding points in R where f has value 1 or -1 confirms you that the absolute maximum and minimum values of f are 1 and -1.)

(7) (4.2.46(b)) Solving $\nabla f(x, y) = (3ye^x - 3e^{3x}, 3e^x - 3y^2) = (0, 0)$, we get $e^x = y^2, 3y^3 - 3y^2 = 0$. So $(0, 1)$ is the only critical point. $Hf(0, 1) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$ is negative definite, hence $(0, 1)$ is a local maximum. However, let us fix $x = 0$ and send y to the negative infinity, then $\lim_{y \to -\infty} f(0, y) = \lim_{y \to -\infty} 3y - 1 - y^3 = \infty$. Therefore f does not have a global maximum.

(8) (i) Using Lagrange multiplier method, we get $\begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} = \lambda \begin{pmatrix} 2(y+z) \\ 2(z+x) \\ 2(x+y) \end{pmatrix}$. So $(y - x)z = 2\lambda(y - x)$. If $x \neq y$ then $z = 2\lambda$, so $2\lambda y = 2\lambda(y + 2\lambda)$, and $2\lambda = z = 0$, and $xy = 0$, this is impossible since $a \neq 0$. So $x = y$. Similarly repeat this argument, and we get $x = y = z$. So $6x^2 = a$ implies $(x, y, z) = (\sqrt{\frac{a}{6}}, \sqrt{\frac{a}{6}}, \sqrt{\frac{a}{6}})$ is the only critical point.

(ii) Without loss of generality, let $x < \frac{\sqrt{a}}{3\sqrt{6}}$ at Q. Since $yz < xy + yz + xz = \frac{a}{2}$, it implies that $V(Q) = xyz < \frac{\sqrt{a}}{3\sqrt{6}} \cdot \frac{a}{2} = (\frac{a}{6})^{3/2} = V(P)$

(iii) K is defined by closed relations, hence it is closed. To prove that K is bounded, notice that $\frac{a}{2} = xy + yz + zx = x(y + z) + yz > x(y + z) \geq \frac{2\sqrt{a}}{3\sqrt{6}}x$. Hence x is bounded above as well as below. Similarly y, z are also bounded. Hence K is contained in a bounded box, hence K is bounded.

(iv) Since K is compact, there exists a maximum point of V. By (i), we know that V has the only critical point P. To see the values of V on the boundaries of K, let $x = \frac{\sqrt{a}}{3\sqrt{6}}$ without loss of generality. Since $yz < xy + yz + xz = \frac{a}{2}$, we have $xyz = \frac{\sqrt{a}}{3\sqrt{6}} yz < (\frac{a}{6})^{3/2} = V(P)$. Hence the value of V on the boundary is always less than $V(P)$. Therefore V has the maximal value on K at P.

(v) By (ii), we know that V has smaller value than $V(P)$ at any point outside of K. Therefore V has the maximal value on A at P.

(9) (4.3.2) $\nabla f(x, y) = (0, 1) = \lambda \nabla g(x, y) = \lambda(4x, 2y)$. $\therefore (x, y) = (0, 2), (0, -2)$.

(10) (4.3.8) $(1, 1, 1) = \lambda(-2x, 2y, 0) + \mu(1, 0, 2)$. So $\mu = 1/2$, $2\lambda y = 1$, $-2\lambda x + \mu = 1$. Therefore $\lambda = \pm\sqrt{3}/4$ and $(x, y, z) = (-1/\sqrt{3}, 2/\sqrt{3}, (1 + 1/\sqrt{3})/2), (1/\sqrt{3}, -2/\sqrt{3}, (1 - 1/\sqrt{3})/2)$.

(11) (4.3.18) Since the sphere is closed and bounded, it is compact. Hence there must be maximum and minimum points. By Lagrange multiplier method, we have $(1, 1, -1) = \lambda(2x, 2y, 2z)$, hence $x = y = -z$. From $3x^2 = 81$, we get two critical points $(x, y, z) = (3\sqrt{3}, 3\sqrt{3}, -3\sqrt{3}), (-3\sqrt{3}, -3\sqrt{3}, 3\sqrt{3})$. At each point, the value of f is $9\sqrt{3}$ and $-9\sqrt{3}$. These are the maximum and minimum values.